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CHAPTER ONE: LINEAR PROGRAMMING AND MODELS OF PRACTICAL PROBLEMS

1.1. Definition of optimization and its classification

Optimization is the selection of a best element or solution of a problem (with regard to some criterion) from some set of available alternatives. Optimization is also called mathematical programming.

Moreover, optimization (mathematical programming) is:

- ❖ the process of obtaining the ‘best’, if it is possible to measure and change what is ‘good’ or ‘bad’.
- ❖ concerned with finding the best (optimal) solution of a problem.
- ❖ defined as the process of finding the conditions that give the maximum or minimum value of a function.
- ❖ the act of obtaining the best result under given circumstance.

In optimization one investigates problems of the determination of a minimal point of a functional on a nonempty subset of a real linear space. To be more specific this means: Let X be a real linear space, let S be a nonempty subset of X , and let $f: S \rightarrow \mathbb{R}$ be a given functional. We ask for the minimal points of f on S . An element $\bar{x} \in S$ is called a minimal point of f on S if

$$f(\bar{x}) \leq f(x) \text{ for all } x \in S.$$

The set S is also called constraint set and the functional f is called objective functional. Moreover, the constraint set S is also called feasible set or feasible region and the function f is called objective function.

Mathematically, general optimization (mathematical programming) is defined as:

$$\text{Optimize } f(x) \text{(2.1)}$$

The expression in (2.1) is called general optimization (mathematical programming) problem and in the problem to optimize means that either to maximize or to minimize.

In particular, maximize $f(x)$ is a maximization optimization problem and minimize $f(x)$ is a minimization optimization problem.

In optimization problem one seeks to maximize or minimize a specific quantity, called the objective, which depends on a finite number of input variables. These variables may be independent of one another, or they may be related through one or more constraints.

In this course, we are interested in problems in which S is a subset of finite dimensional real space $\mathbb{R}^n$ and a real valued function f . Frequently, the definition of S involves system of equations and inequalities, which we call constraints.

The simplest case is when $S = \mathbb{R}^n$ and such a problem is called the unconstrained optimization problem. If S is a strict subset of the space $\mathbb{R}^n$ ($S \subset \mathbb{R}^n$), we call the problem is constrained optimization problem.

Generally, based on the functions involved in the optimization problem (in the objective and constraint set) it can be divided in to linear and Nonlinear optimization problem.

- If the objective function f is linear function and the constraint set S is defined by finitely many linear equations and inequalities, the problem is called linear optimization problem (LOP) or linear programming problem (LPP).
- If the objective function f or some of the equations or inequalities defining the feasible set S are nonlinear, the optimization problem is called the nonlinear optimization (or nonlinear programming) problem.

A mathematical programming is a problem in which the objective and constraints are given as mathematical functions and functional relationships. Mathematically, optimization (mathematical programming) is defined as:

$$\begin{aligned} & \text{Optimize } z = f(x_1, x_2, \dots, x_n) \\ & \text{Subject to } \left. \begin{array}{l} g_1(x_1, x_2, \dots, x_n) \\ g_2(x_1, x_2, \dots, x_n) \\ \vdots \\ g_m(x_1, x_2, \dots, x_n) \end{array} \right\} \begin{array}{l} \leq \\ = \\ \geq \end{array} \left\{ \begin{array}{l} b_1 \\ b_2 \\ \vdots \\ b_m \end{array} \right. \end{aligned} \quad (2.2)$$

Each of the m constraint relationships in (2.2) involves one of the three signs $\leq, \geq, =$. Unconstrained mathematical programs are covered by the formalism (2.2) if each function g_i , for $i = 1, 2, \dots, m$, is chosen as zeros and each constant b_i ($i = 1, 2, \dots, m$) is chosen as zero.

A mathematical programming (2.2.) is linear if $f(x_1, x_2, \dots, x_n)$ and each $g_i(x_1, x_2, \dots, x_n)$ ($i = 1, 2, \dots, m$) are linear in each of their arguments, that is, if

$$f(x_1, x_2, \dots, x_n) = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad (2.3)$$

$$\text{and } g_i(x_1, x_2, \dots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \quad (2.4)$$

where c_j and a_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) are known constants.

In addition to this, Linear programming (LP) is the problem of optimizing (maximizing or minimizing) a linear function subject to linear constraints. A wide variety of practical problems, from nutrition, transportation, production planning, finance, personal scheduling, and many more areas can be modeled as linear programs.

Therefore, the basic mathematical definition of linear programming problem is

$$\max/\min z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \{\leq, \geq, =\}b_1$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \{\leq, \geq, =\}b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \{\leq, \geq, =\}b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

Here $c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is the objective function (or criterion function) to be minimized or maximized. The coefficients $c_1, c_2, \dots, c_n$ are known as cost coefficients and $x_1, x_2, \dots, x_n$ are the decision variables to be determined. The constraints $x_1, x_2, \dots, x_n \geq 0$ are the non-negativity constraints.

Note:

1. $f(x) \rightarrow \max, x \in S$ is a maximization problem (i.e., the maximum value of f in the set S).
2. $f(x) \rightarrow \min, x \in S$ is a minimization problem (i.e., find the value $x^* \in S$ such that $f(x^*) \leq f(x), \forall x \in S$).
3. The set S is called feasible set or constraint set.
4. Each element of S is called feasible solution.
5. The function f is called objective function of the problem (i.e., the function to be optimized is called objective function).
6. If $x^* \in S$ such that either $f(x^*) \leq f(x), \forall x \in S$ or $f(x^*) \geq f(x), \forall x \in S$, then x^* is called optimal(best) solution or optimal point.

$$\triangleright f(x^*) \geq f(x), \forall x \in S \Leftrightarrow f(x^*) = \max_{x \in S} f(x).$$

$$\triangleright f(x^*) \leq f(x), \forall x \in S \Leftrightarrow f(x^*) = \min_{x \in S} f(x).$$

- $\triangleright$ The value of objective function at the optimal solution, $f(x^*)$, is called optimal value.

Remark:

1. A minimization optimization problem can be converted to maximization optimization problem and vice versa.

Proof: Exercise!

1.2. Models and Model building in optimization

*Models are abstractions of reality and/ or representation of a particular thing, idea, or condition. A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed as mathematical modeling.

Mathematical models are used in the natural sciences and engineering disciplines, as well as in the social sciences. Physicists, engineers, statisticians, operations research analysts, and economists use mathematical models most extensively.

*In general, a model may help:

- To explain a system in a simple way,
- To study the effects of different components and
- To make predictions about behavior.

*Moreover, a mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms.

Mathematical models are also idealized representations, but they are expressed in terms of mathematical symbols and expressions. Similarly, the mathematical model of a business problem is the system of equations and related mathematical expressions that describe the essence of the problem.

Although we refer to the mathematical model of a business problem, real problems normally don't have just a single "right" model. It is even possible that two or more completely different types of models may be developed to help analyze the same problem.

Mathematical models have many advantages over a verbal description of the problem. One advantage is that a mathematical model describes a problem much more concisely. This tends to make the overall structure of the problem more comprehensible, and it helps to reveal important cause-and-effect relationships. In this way, it indicates more clearly what additional data are relevant to the analysis. It also facilitates dealing with the problem in its entirety and considering all its interrelationships simultaneously. Finally, a mathematical model forms a bridge to the use of high-powered mathematical techniques and computers to analyze the problem. Indeed, packaged software for both personal computers and mainframe computers has become widely available for solving many mathematical models.

Optimization models attempt to express, in mathematical terms, the goal of solving a problem in the "best" way. That might mean running a business to maximize profit, minimize loss, maximize efficiency, or minimize risk. It might mean designing a bridge to minimize weight or maximize strength. It might mean selecting a flight plan for an aircraft to minimize time or fuel use. The desire to solve a problem in an optimal way is so common that optimization models arise in almost every area of application.

In solving real life problems mathematically, we should follow the following procedures:

1. Define the problem of interest and gather relevant data.

The first order of real life problem is to study the relevant system in detail and develop a well-defined statement of the problem to be considered. This includes determining such things as the appropriate objectives, constraints on what can be done, interrelationships between the areas to be

studied and other areas of the organization, possible alternative courses of action, and time limits for making a decision, and so on. This process of problem definition is a crucial one because it greatly affects how relevant the conclusions of the study will be. It is difficult to extract a “right” answer from the “wrong” problem!

Moreover, collection of relevant data, and identification of system constraints, restrictions or limitations and selection of objective of the problem, parameter and decision variable are the basic execution of this step.

2. Formulate a mathematical model to represent the problem.

After the decision maker’s problem is defined, the next phase is to reformulate this problem in a form that is convenient for analysis. The conventional optimization problem (OP) approach for doing this is to construct a mathematical model that represents the essence of the problem. Before discussing how to formulate such a model, we first explore the nature of models in general and of mathematical models in particular.

In general, in this stage the set of abstract mathematical formulae will be used to represent the problem under study.

A crucial step in formulating an OP model is the construction of the objective function. This requires developing a quantitative measure of performance relative to each of the decision maker’s ultimate objectives that were identified while the problem was being defined.

3. Develop a computer-based procedure for deriving solutions to the problem from the model.

After a mathematical model is formulated for the problem under consideration, the next phase in real life problem study is to develop a procedure (usually a computer-based procedure) for deriving solutions to the problem from this model. You might think that this must be the major part of the study, but actually it is not in most cases. Sometimes, in fact, it is a relatively simple step, in which one of the standard **algorithms** (systematic solution procedures) of the problem is applied on a computer by using one of a number of readily available software packages.

A common theme in optimization problem is the search for an **optimal**, or **best**, **solution**. Indeed, many procedures have been developed, and are presented, for finding such solutions for certain kinds of problems. However, it needs to be recognized that these solutions are optimal only with respect to the model being used. Since the model necessarily is an idealized rather than an exact representation of the real problem, there cannot be any utopian guarantee that the optimal solution for the model will prove to be the best possible solution that could have been implemented for the real problem. There just are too many imponderables and uncertainties associated with real problems. However, if the model is well formulated and tested, the resulting solution should tend to be a good approximation to an ideal course of action for the real problem. Therefore, rather than be deluded into demanding the impossible, you should make the test of the practical success of an

optimization problem study hinge on whether it provides a better guide for action than can be obtained by other means.

4. Test the model and refine it as needed (Model testing, analysis, and restructuring).

Developing a large mathematical model is analogous in some ways to developing a large computer program. When the first version of the computer program is completed, it inevitably contains many bugs. The program must be thoroughly tested to try to find and correct as many bugs as possible. Eventually, after a long succession of improved programs, the programmer (or programming team) concludes that the current program now is generally giving reasonably valid results. Although some minor bugs undoubtedly remain hidden in the program (and may never be detected), the major bugs have been sufficiently eliminated that the program now can be reliably used.

Similarly, the first version of a large mathematical model inevitably contains many flaws. Some relevant factors or interrelationships undoubtedly have not been incorporated into the model, and some parameters undoubtedly have not been estimated correctly. This is inevitable, given the difficulty of communicating and understanding all the aspects and subtleties of a complex operational problem as well as the difficulty of collecting reliable data. Therefore, before you use the model, it must be thoroughly tested to try to identify and correct as many flaws as possible. Eventually, after a long succession of improved models, the real problem team concludes that the current model now is giving reasonably valid results. Although some minor flaws undoubtedly remain hidden in the model (and may never be detected), the major flaws have been sufficiently eliminated that the model now can be reliably used.

This process of testing and improving a model to increase its validity is commonly referred to as **model validation**. It is difficult to describe how model validation is done, because the process depends greatly on the nature of the problem being considered and the model being used.

5. Implementation.

After a system is developed for applying the model, the last phase of a problem study is to implement this system as prescribed by user. This phase is a critical one because it is here, and only here, that the benefits of the study are reaped. Therefore, it is important for the optimization problem team to participate in launching this phase, both to make sure that model solutions are accurately translated to an operating procedure and to rectify any flaws in the solutions that are then uncovered.

In particular, a linear programming (LP) model is characterized by proportionality, divisibility, additivity, and certainty.

- **Proportionality** means that the contribution toward the objective function and constraints of decisions are directly proportional to its values, i.e., there are no economies of scale such as quantity-based discounts. In particular, this means that decision variables in an LP

are raised to the first power only. So terms of the form $c_i x_i^{\frac{1}{2}}$ or $c_i x_i^2$ (where c_i is a coefficient and x_i a decision variable) are not permitted in an LP.

- **Divisibility** means that the decision variables can take on any real number (integer, rational or irrational).
- **Additivity** means that the contribution of a decision variable toward the objective or constraints does not depend on other decision variables. In other words, the total contribution is the sum of individual contributions of each decision variable.
- **Certainty** means that the data used as coefficients for a linear programming model, such as the objective coefficients C and constraint coefficients A and b , are known with certainty. For example, it is assumed in the diet model that the cost of one serving of bananas is 10 cents and that one serving of peanut butter provides 130 grams of fat, and these values are assumed as correct or valid for the model.

*In general the relation between real problem, mathematical model, solving the model and interpretation can describe as the diagram below.

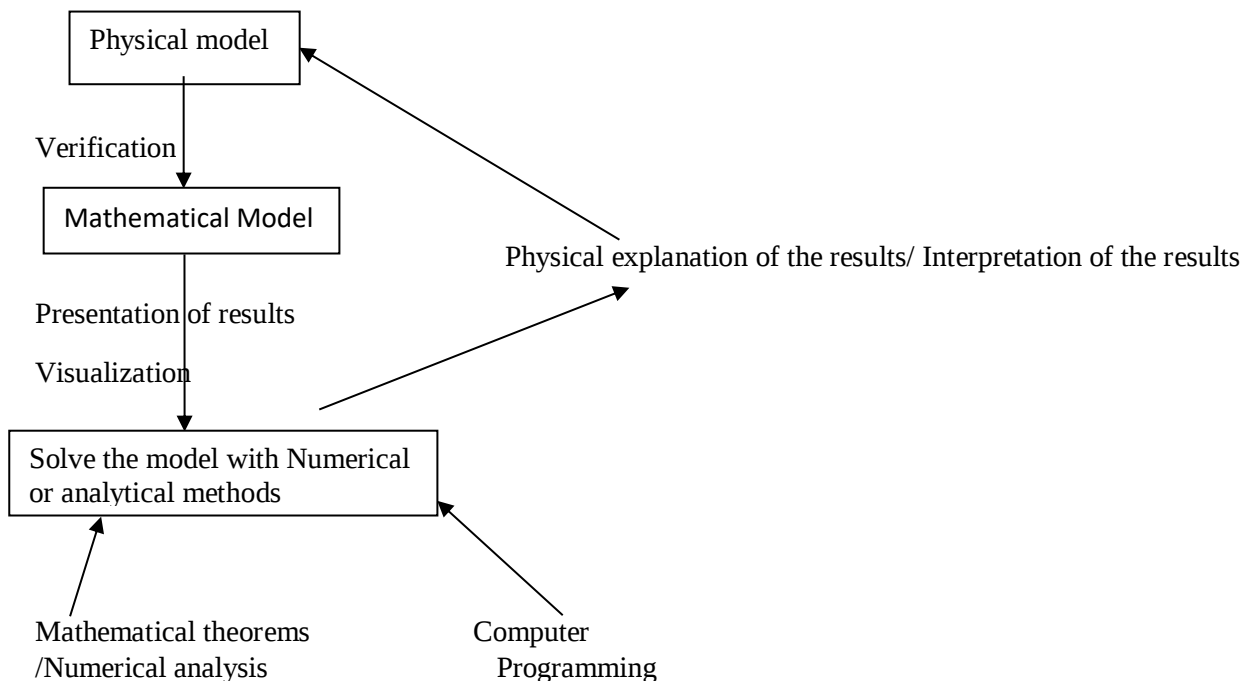


Fig. 2.1 Relationship of real problem, Mathematical model and numerical/analytical method

1.3. Some examples of practical problems

- Diet problem
- Production planning problems
- Transportation problems
- Assignment problems

Example 1. (Diet problem)

Due to a limited budget you would like to find the most economical mix of food items subject to providing sufficient daily nutrition. The available food items are peanut butter, bananas, and chocolate, and the cost per serving is 20 cents, 10 cents, and 15 cents, respectively. The amount of nutrients of fat, carbohydrates, and protein per serving of peanut butter is 130 grams, 51.6 grams, and 64.7 grams. For bananas, the amounts of these nutrients per serving are 1 gram, 51 grams, and 2 grams, whereas for chocolate the amounts per serving are 12 grams, 22 grams, and 2 grams. Suppose it is decided by a dietician that you need at least 35 grams of fat, at least 130 grams of carbohydrate, and at least 76 grams of protein daily. Then, a combination of these three food items should provide you with sufficient amounts of each nutrient for the day. Formulate a mathematical programming diet problem to find the least cost of the food items that provide a sufficient amount of nutrients.

Solution: The problem of determining the least cost combination of the food items that provide a sufficient amount of nutrients can be represented as a problem of the following form:

Minimize **cost of food items used**
Subject to **food items must provide enough fat**
 food items must provide enough carbohydrates
 food items must provide enough protein

This form of the problem reveals two major components, i.e.,

- i) there is a goal or objective (minimize cost of food items used) and
- ii) a set of constraints that represent the requirements for the problem (food items must provide enough nutrition). A linear programming problem will exhibit the same form, but with mathematical representations of the components of the problem.

In general, a linear program will consist of decision variables, an objective, and a set of constraints. To illustrate the formulation of the diet problem as a linear program, let

x_{pb} = serving of peanut butter

x_b = serving of bananas

x_c = serving of chocolate

These variables represent the quantities of each food item to be used and should be non-negative, i.e., $x_{pb} \geq 0, x_b \geq 0$ and $x_c \geq 0$, and are called decision variables since they must be determined.

The total cost associated with any particular combination of food items, x_{pb}, x_b and x_c is,

$$z = 0.20x_{pb} + 0.10x_b + 0.15x_c \dots\dots\dots(2.2.1)$$

Which says that the total cost of food items used is the sum of the costs incurred from the use of each food item.

To ensure that the mix of the three foods will have enough fat one can impose the following linear inequality constraint

$$130x_{pb} + x_b + 12x_c \geq 35, \dots\dots\dots(2.2.2)$$

Which expresses that, the total amount of fat from the three food items should be at least the required minimum of 35 grams.

Similarly, one can impose the inequality

$$51.6x_{pb} + 51x_b + 22x_c \geq 130, \dots\dots\dots(2.2.3)$$

to ensure that a combination of food items will have enough carbohydrates and finally impose the constraint

$$64.7x_{pb} + 2x_b + 2x_c \geq 76, \dots\dots\dots(2.2.4)$$

to ensure enough protein.

Then, the diet problem linear program can be expressed as

$$\begin{aligned} \text{minimize } Z &= 0.20x_{pb} + 0.10x_b + 0.15x_c \\ \text{subject to } &130x_{pb} + x_b + 12x_c \geq 35 \\ &51.6x_{pb} + 51x_b + 22x_c \geq 130 \dots\dots\dots(2.2.5) \\ &64.7x_{pb} + 2x_b + 2x_c \geq 76 \\ &x_{pb}, x_b, x_c \geq 0 \end{aligned}$$

The problem can be interpreted as the problem of determining non-negative values of x_{pb} , x_b and x_c so as to minimize the (cost) function $Z = 0.20x_{pb} + 0.10x_b + 0.15x_c$ subject to meeting the nutritional requirements as embodied in the constraints (2.2.2)—(2.2.4). The function $Z = 0.20x_{pb} + 0.10x_b + 0.15x_c$ is called the objective function and the objective is the minimization of this function. It is important to observe that the objective function and the left hand sides of the constraints are all linear, i.e., all variables are taken to the power of one.

The diet problem highlights the major components of a linear programming problem where an LP problem consists of decision variables, an objective (optimize a linear objective function), linear constraints and possibly some sign restrictions on variables. The diet problem can be generalized to where there are n types of food items and m nutritional requirements, where x_i represents the number of serving of food item i ($i = 1, 2, \dots, n$), c_i is the cost of one serving of food item i ($i = 1, 2, \dots, n$), a_{ij} is the amount of nutrient i in one serving of food item j ($j = 1, 2, \dots, m$) and b_j is the minimum amount of nutrient j required ($j = 1, 2, \dots, m$). Then, the general formulation of the diet problem can be written as

$$\begin{aligned} \text{minimize } z &= c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{Subject to } &a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m \\ &x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

In matrix form, the general diet problem can be represented as;

$$\begin{aligned} \text{minimize } z &= c^t x \\ \text{subject to } Ax &\geq b, \quad x \geq 0 \end{aligned}$$

Where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the vector of food items,

$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{bmatrix} a_1^t \\ \vdots \\ a_m^t \end{bmatrix}$ is a matrix of dimension $m \times n$ and a_j^t is an n -dimensional

row vector that represents the j^{th} row of A , i.e., the total amount of nutrient j that would be obtained from the amount of food items given by x ,

$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is the vector of costs of food items, and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is the vector of nutrition

requirements.

For example, for the diet problem (2.2.5) we have $n = 3$ and $m = 3$ with

$$A = \begin{pmatrix} 130 & 1 & 12 \\ 51.6 & 51 & 22 \\ 64.7 & 2 & 2 \end{pmatrix}, c = \begin{pmatrix} 0.20 \\ 0.10 \\ 0.15 \end{pmatrix}, x = \begin{pmatrix} x_{pb} \\ x_b \\ x_c \end{pmatrix} \text{ and } b = \begin{pmatrix} 35 \\ 130 \\ 76 \end{pmatrix}.$$

The objective function is $c^t x = (0.20 \quad 0.10 \quad 0.15) \begin{pmatrix} x_{pb} \\ x_b \\ x_c \end{pmatrix} = 0.20x_{pb} + 0.10x_b + 0.15x_c$.

The first constraint is $a_1^t x = [130 \quad 1 \quad 2] \begin{bmatrix} x_{pb} \\ x_b \\ x_c \end{bmatrix} = 130x_{pb} + x_b + 2x_c \geq b_1 = 35$.

The second constraint $a_2^t x = [51.6 \quad 51 \quad 22] \begin{bmatrix} x_{pb} \\ x_b \\ x_c \end{bmatrix} = 51.6x_{pb} + 51x_b + 22x_c \geq b_2 = 130$.

The third constraint is $a_3^t x = [64.7 \quad 2 \quad 2] \begin{bmatrix} x_{pb} \\ x_b \\ x_c \end{bmatrix} = 64.7x_{pb} + 2x_b + 2x_c \geq b_3 = 76$.

Example 2. (Transportation problem)

Reactiveeno Inc. is a company that produces a special type of facial cream that aids in reducing acne. This product is manufactured in n plants across North America. Every month the facial cream product is shipped from the n plants to m warehouses. Plant i has a supply of u_i units of the product. Warehouse j has a demand of d_j units. The cost of shipping one unit from plant i to warehouse j is c_{ij} . The problem of finding the least-cost shipping pattern from plants to warehouses can be formulated as LP.

Solution: Let x_{ij} be the number of units of product shipped from plant i to warehouse j . so the objective is to minimize the cost of shipping over all possible plant warehouse pairs. There are two classes of constraints. One class of constraints must ensure that the total amount shipped from a plant i to warehouse j , i.e., $\sum_{j=1}^m x_{ij}$ does not exceed the supply u_i of plant i , whereas the other class of constraints ensures that the demand d_j of a warehouse j is met from the total amount

$\sum_{i=1}^n x_{ij}$ shipped by plants. Then, the transportation problem can be formulated as the following LP.

$$\begin{aligned} & \text{Minimize } \sum_{i=1}^n \sum_{j=1}^m c_{ij} x_{ij} \\ & \text{Subject to } \sum_{j=1}^m x_{ij} \leq u_i, i = 1, \dots, n \\ & \sum_{i=1}^n x_{ij} \geq d_i, j = 1, \dots, m \\ & x_{ij} \geq 0, i = 1, \dots, n, j = 1, 2, \dots, m \end{aligned}$$

Example 3. (Production planning)

Consider a company that produces n different products. Each product uses m different resources. Suppose that resources are limited and the company has only b_i units of resources i available for each $i = 1, 2, \dots, m$. Further, each product j requires a_{ij} units of resources i for production. Each unit of product j made generate revenue of p_j dollars. The company wishes to find a production plan that is the quantity of each product produce that maximize revenue. Formulate a mathematical model of LPP.

Solution: the problem can be formulated as linear program. The first step is to define the decision variables. Let x_j be a decision variable that represents the amount of product j produced by the company. Suppose that fractional amounts of a product are allowed and this amount should be non-negative. Since a negative value for x_j is meaningless. Then a production plan is represented by a vector $x = [x_1, x_2, \dots, x_n]^T$. The contribution toward revenue from the production of x_j units of product j is $p_j x_j$ and so the total revenue from a production plan x is then $p_1 x_1 + p_2 x_2 + \dots + p_n x_n = \sum_{j=1}^n p_j x_j$. the contribution toward using resources i from the production of x_j units of product j is $\sum_{j=1}^n a_{ij} x_j$ and this quantity can be exceed by b_i . Since total revenue to maximized and resources limitation must be observed over all resources, then the LPP is

$$\begin{aligned} & \text{Maximize } \sum_{j=1}^n p_j x_j \\ & \text{subject to } \sum_{j=1}^n a_{ij} x_j \leq b_i, i = 1, 2, \dots, m \\ & x_j \geq 0, j = 1, 2, \dots, n \end{aligned}$$

Example 4. (Assignment problem)

A special case of the transportation problem above is when there are as many plants as warehouses, i.e., $m = n$ and each warehouse demands exactly one unit of the product and each plant produces only one unit of the product, i.e., $d_i = u_i = 1$. Then the model takes the following form

$$\begin{aligned} &\text{Minimize } \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ &\text{Subject to } \sum_{j=1}^n x_{ij} = 1, i = 1, \dots, n \\ &\quad \sum_{i=1}^n x_{ij} = 1, j = 1, \dots, n \\ &\quad x_{ij} \geq 0, i = 1, \dots, n, j = 1, 2, \dots, n \end{aligned}$$

This special case is known as the assignment problem and has the interpretation of matching persons with jobs so that each person gets one job and each job gets one person. c_{ij} in this context represents the cost of assigning person i to job j . This quantity can reflect the different skill levels of workers,

Example 5. (Product mix problem)

A manufacturing process requires three different inputs; A, B, and C. A sandal soap of the first type requires 30 grams of A, 20 gram of B and 6 gram of C, while this data for the second type of soap is 25, 5 and 15, respectively. The maximum availability of the inputs A, B and C are 6000, 3000, and 3000 grams, respectively. The selling prices of the sandal soap of the first and second type are \$ 14 and 15 respectively. The profit is proportional to the amount of soaps manufactured. How much soap of the first and second kinds should be manufactured to maximize the profit? Assume that the market has unlimited demand. Formulate the mathematical model of the problem.

Solution: left as exercise!

CHAPTER TWO: SOLUTIONS OF LINEAR PROGRAMMING AND SOME BASIC THEOREMS

Introduction

Linear programs can be studied both algebraically and geometrically. The two approaches are equivalent, but one or the other may be more convenient for answering a particular question about a linear program.

The algebraic point of view is based on writing the linear program in a particular way, called *standard form*. Then the coefficient matrix of the constraints of the linear program can be analyzed using the tools of linear algebra. For example, we might ask about the rank of the matrix, or for a representation of its null space. It is this algebraic approach that is used in the simplex method.

The geometric point of view is based on the geometry of the feasible region and uses ideas such as convexity to analyze the linear program. It is less dependent on the particular way in which the constraints are written. Using geometry (particularly in two-dimensional problems where the feasible region can be graphed) makes many of the concepts in linear programming easy to understand, because they can be described in terms of intuitive notions such as moving along an edge of the feasible region.

There is a direct correspondence between these two points of view. This chapter will explore several aspects of this correspondence.

2.1. Polyhedral sets and convex sets

Definition 3.1 A polyhedral set is a set that can be described in the form $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where A is $m \times n$ matrix and b is a vector in $\mathbb{R}^m$.

Definition 3.2 Let P be a polyhedron. A vector $x \in P$ is an extreme (or corner) point of P if we cannot find two vectors $y, z \in P$, both different from x and a scalar $\lambda \in [0, 1]$ such that $x = \lambda y + (1 - \lambda)z$.

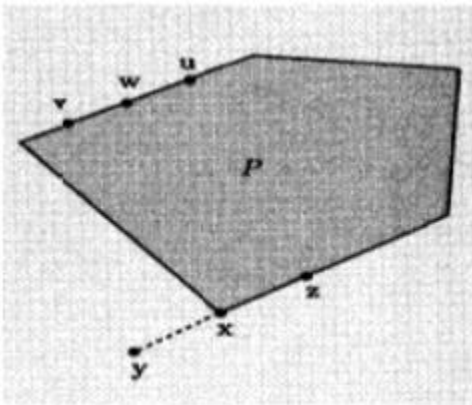


Figure 3.1 The vector w is not extreme point because it is a convex combination of v and u . The vertex x is extreme point; if $x = \lambda y + (1 - \lambda)z$ and $\lambda \in [0, 1]$, then either $y \notin P$, or $z \notin P$ or $x = y$ or $x = z$.

The feasible set of any linear programming problem can be described by inequality constraints of the form $Ax \leq b$, and is therefore a polyhedron. In particular, a set of the form $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ is also a polyhedron and will be referred to as a polyhedron in standard form.

A polyhedron can either “extend to infinity” or can be confined in a finite region. The definition that follows refers this definition.

Definition 3.3 A set $S \subseteq \mathbb{R}^n$ is bounded if there exists a constant K such that the absolute value of every component of every element of S is less than or equal to K .

The next definition deals with polyhedra determined by a single linear constraint.

Definition 3.4 Let a be a nonzero vector in $\mathbb{R}^n$ and let b be a scalar.

- The set $\{x \in \mathbb{R}^n | ax = b\}$ is called a hyperplane.
- The set $\{x \in \mathbb{R}^n | ax \leq b\}$ is called a half space.

Note that a hyper plane is the boundary of a corresponding half space. In addition, the vector a in the definition of the hyper plane is perpendicular to the hyper plane itself. [To see this, note that if x and y belong to the same hyper plane, then $ax = ay$. Hence, $a(x - y) = 0$ and therefore a is orthogonal to any direction vector confined to the hyper plane.] Finally, note that a polyhedron is equal to the intersection of a finite number of half spaces.

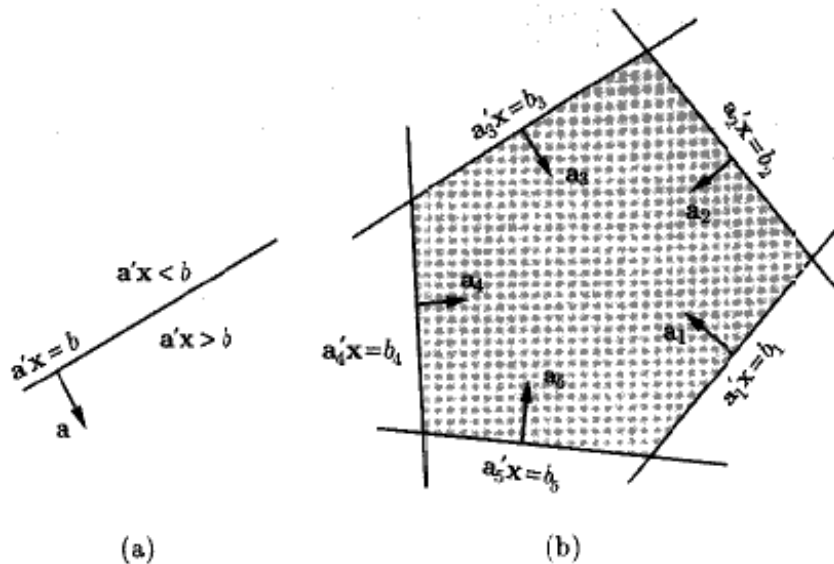


Figure 3.2

- A hyper plane and two half spaces.
- The polyhedron $\{x | a'_i x \geq b_i, i = 1, 2, \dots, 5\}$ is the intersection of five half space. Note that each vector a'_i is perpendicular to the hyperplane $\{x | a'_i x = b_i\}$

Note: The intersection of a finite number of closed half spaces is called a polyhedron (or polyhedral set). A bounded polyhedron is called a polytope. So the feasible set P of any linear program is a polyhedral set.

2.2. Convex set and the corner point theorem

Convex set

Definition 3.2.1 A set $S \subseteq \mathbb{R}^n$ is convex if for any $x, y \in S$ and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$.

Note that if $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y$ is a weighted average of the vectors x, y and therefore belongs to the line segment joining x and y . Thus, a set is convex if the segment joining any two of its elements is contained in the set.

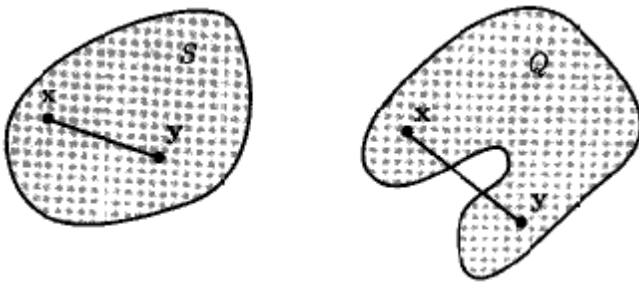


Figure 3.3 The set S is convex, but the set Q is not, because the segment joining x and y is not contained in Q .

Definition 3.2.2 Let $x^1, \dots, x^k$ be vectors in $\mathbb{R}^n$ and let $\lambda_1, \dots, \lambda_k$ be nonnegative scalars whose sum is unity. Then

- The vector $\sum_{i=1}^k \lambda_i x^i$ is said to be a convex combination of the vectors $x^1, \dots, x^k$.
- The convex hull of the vectors $x^1, \dots, x^k$ is the set of all convex combination of these vectors.

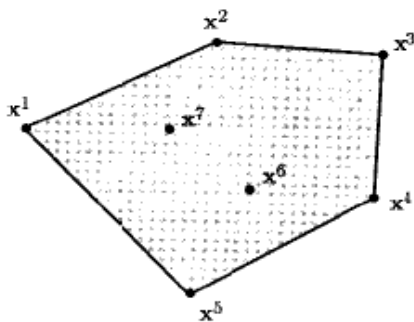


Figure 3.4 The convex hull of seven points in 2-dimensional space.

Theorem 3.1

- The intersection of convex sets is convex.
- Every polyhedron is a convex set.
- A convex combination of a finite number of elements of a convex set also belongs to that set.
- The convex hull of a finite number of vectors is a convex set.

Proof:

a) Suppose there is an arbitrary collection of convex sets S_i indexed by the set I . Consider the intersection $\cap_{i \in I} S_i$ and let x and y be elements of this intersection. For any $\lambda \in [0, 1]$, then $z = \lambda x + (1 - \lambda)y$ is in every set S_i since x and y are in S_i for every $i \in I$ and S_i is a convex set. Thus, $\cap_{i \in I} S_i$ is a convex set. ■

- (b) Let $\mathbf{a}$ be a vector and let b a scalar. Suppose that $\mathbf{x}$ and $\mathbf{y}$ satisfy $\mathbf{a}'\mathbf{x} \geq b$ and $\mathbf{a}'\mathbf{y} \geq b$, respectively, and therefore belong to the same halfspace. Let $\lambda \in [0, 1]$. Then, $\mathbf{a}'(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) \geq \lambda b + (1 - \lambda)b = b$, which proves that $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ also belongs to the same halfspace. Therefore a halfspace is convex. Since a polyhedron is the intersection of a finite number of halfspaces, the result follows from part (a).
- (c) A convex combination of two elements of a convex set lies in that

set, by the definition of convexity. Let us assume, as an induction hypothesis, that a convex combination of k elements of a convex set belongs to that set. Consider $k + 1$ elements $\mathbf{x}^1, \dots, \mathbf{x}^{k+1}$ of a convex set S and let $\lambda_1, \dots, \lambda_{k+1}$ be nonnegative scalars that sum to 1. We assume, without loss of generality, that $\lambda_{k+1} \neq 1$. We then have

$$\sum_{i=1}^{k+1} \lambda_i \mathbf{x}^i = \lambda_{k+1} \mathbf{x}^{k+1} + (1 - \lambda_{k+1}) \sum_{i=1}^k \frac{\lambda_i}{1 - \lambda_{k+1}} \mathbf{x}^i. \quad (2.1)$$

The coefficients $\lambda_i/(1 - \lambda_{k+1})$, $i = 1, \dots, k$, are nonnegative and sum to unity; using the induction hypothesis, $\sum_{i=1}^k \lambda_i \mathbf{x}^i / (1 - \lambda_{k+1}) \in S$. Then, the fact that S is convex and Eq. (2.1) imply that $\sum_{i=1}^{k+1} \lambda_i \mathbf{x}^i \in S$, and the induction step is complete.

- (d) Let S be the convex hull of the vectors $\mathbf{x}^1, \dots, \mathbf{x}^k$ and let $\mathbf{y} = \sum_{i=1}^k \zeta_i \mathbf{x}^i$, $\mathbf{z} = \sum_{i=1}^k \theta_i \mathbf{x}^i$ be two elements of S , where $\zeta_i \geq 0$, $\theta_i \geq 0$, and $\sum_{i=1}^k \zeta_i = \sum_{i=1}^k \theta_i = 1$. Let $\lambda \in [0, 1]$. Then,

$$\lambda \mathbf{y} + (1 - \lambda) \mathbf{z} = \lambda \sum_{i=1}^k \zeta_i \mathbf{x}^i + (1 - \lambda) \sum_{i=1}^k \theta_i \mathbf{x}^i = \sum_{i=1}^k (\lambda \zeta_i + (1 - \lambda) \theta_i) \mathbf{x}^i.$$

We note that the coefficients $\lambda \zeta_i + (1 - \lambda) \theta_i$, $i = 1, \dots, k$, are nonnegative and sum to unity. This shows that $\lambda \mathbf{y} + (1 - \lambda) \mathbf{z}$ is a convex combination of $\mathbf{x}^1, \dots, \mathbf{x}^k$ and, therefore, belongs to S . This establishes the convexity of S . □

Note: The union of Convex set is not necessarily convex. For instance, the sets $A = \{(x, y) \in \mathbb{R}^2 | x + y \leq 1, x \geq 0, y \geq 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 | 0 \leq -y \leq 1, 1 \geq -x \geq 0\}$ are Convex sets, but $A \cup B$ is not convex (Take $z = \lambda(0, 1) + (1 - \lambda)(-0.5, -0.5)$ for $\lambda = \frac{1}{2}$, $z \notin A \cup B$).

Theorem 3.2: If a linear programming problem has an optimal solution, then it has an optimal solution on a corner point of the feasible region.

Proof: Suppose that $z^* = \mathbf{c}\mathbf{x}$ is an optimal solution. If $\mathbf{x}$ is a corner point, then we are done. Suppose that $\mathbf{x}$ is not a corner point, then there are corner points of the feasible region such that there are $\mathbf{x}_i$ and α_i that $\sum \alpha_i \mathbf{x}_i = \mathbf{x}$ and $\sum \alpha_i = 1$. Then $\mathbf{c}\mathbf{x} = \mathbf{c}(\sum \alpha_i \mathbf{x}_i) = \sum \mathbf{c}\alpha_i \mathbf{x}_i = \sum \alpha_i \mathbf{c}\mathbf{x}_i \leq \sum \alpha_i z^* \leq z^*$. Since $z^* = \mathbf{c}\mathbf{x}$, every step here is an equality. So $\mathbf{c}\mathbf{x}_i = z^*$ for all i . Also this shows that $\mathbf{x}$ is only optimal if it is a convex combination of optimal corner point solutions.

Theorem 3.3 The feasible region of a linear programming problem has a finite number of corner points.

Proof: Suppose that there are n equations and $n + m$ variables. Then there are at most $\binom{m+n}{n}$

ways to choose that n -linearly independent columns of the coefficient matrix A of the LPP.

Theorem 3.4 If a corner point has no adjacent corner point that has better objective function value, then there is no other corner point with better objective value.

Proof: Left as an exercise!

A linear programming problem (LPP) can be solving using;

- A) Graphical method or
- B) Simplex method

2.3. Graphical method

We describe here a geometric procedure for solving a linear programming problem. Even though this method is only suitable for very small problems, it provides a great deal of insight into the linear programming problem.

The forgoing process is convenient for problems with two variables and is obviously impractical for problems with more than three variables.

In general, there two graphical (geometrical) methods of solving a LPP;

- i) Corner point method
- ii) Iso-profit (or Iso-cost) method

Basic procedures of corner point method:

Let $Z = ax + by$ and S is a feasible region of an LPP. Then,

- Sketch the graph of the feasible region S
- Determine all extreme points of S
- Evaluate the objective function, Z , at each extreme point.
- Choose the extreme point at which Z is largest (smallest) for maximization (or minimization) problem. That is, the largest value is maximum value and the smallest is minimum value for the problem.

Basic procedures of Iso-profit (Iso-cost) method:

Let $Z = c_1x_1 + c_2x_2$ and S is a feasible region of an LPP. Then,

- Sketch the graph of the feasible region S in $x_1 - x_2$ plane.

- If S is nonempty, choose $\bar{x} = (\bar{x}_1, \bar{x}_2) \in S$ and let $f(\bar{x}) = k$.
Draw the graph of $f(\bar{x}) = k$ or $c_1x_1 + c_2x_2 = k$. The straight line $c_1x_1 + c_2x_2 = k$ is called Iso-profit line for the maximization problem and Iso-cost for minimization problem. It is also called contour line or objective line or z-line.
- Translate the z-line parallel to itself in the direction of improvement of z until it reaches the last intersection with S .

To be more specific, consider the following problem.

$$\begin{aligned} &\text{Minimize} && \mathbf{c}\mathbf{x} \\ &\text{Subject to} && \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ &&& \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Note that the feasible region consists of all vectors $\mathbf{x}$ satisfying $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Among all such points we wish to find a point with minimal $\mathbf{c}\mathbf{x}$ value. Note that points with the same objective z satisfy the equation $\mathbf{c}\mathbf{x} = z$, that is, $\sum_{j=1}^n c_j x_j = z$. Since z is to be minimized, then the plane (line in a two-dimensional space) $\sum_{j=1}^n c_j x_j = z$ must be moved parallel to itself in the direction that minimizes the objective most. This direction is $-\mathbf{c}$, and hence the plane is moved in the direction $-\mathbf{c}$ as much as possible. This process is illustrated in Figure 3.5. Note that as the optimal point $\mathbf{x}^*$ is reached, the line $c_1x_1 + c_2x_2 = z^*$, where $z^* = c_1x_1^* + c_2x_2^*$, cannot be moved farther in the direction $-\mathbf{c} = (-c_1, -c_2)$ because this will lead to only points outside the feasible region. We therefore conclude that $\mathbf{x}^*$ is indeed the optimal solution. Needless to say, for a maximization problem, the plane $\mathbf{c}\mathbf{x} = z$ must be moved as much as possible in the direction $\mathbf{c}$.

It is worth noting that the optimal point $\mathbf{x}^*$ in figure 3.5 is one of the five corner points that are called extreme points. The broken lines are called Iso-profit lines of the problem.

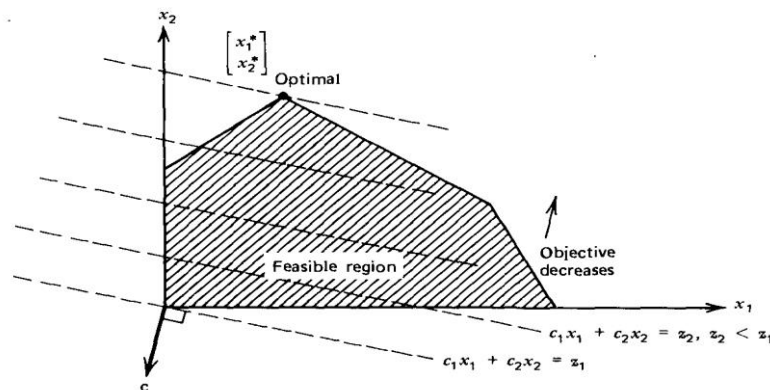


Figure 3.5 Geometric solution

Example 1.

$$\text{Minimize} \quad -x_1 - 3x_2$$

$$\text{Subject to} \quad x_1 + x_2 \leq 6$$

$$-x_1 + 2x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

The feasible region is illustrated in Figure 3.6. The first and second constraints represent points “below” lines 1 and 2 respectively. The nonnegativity constraints restrict the points to be in the first quadrant. The equations $-x_1 - 3x_2 = z$ are called the *objective contours* and are represented by dotted lines in Figure 1.4. In particular the contour $-x_1 - 3x_2 = z = 0$ passes through the origin. The contours are moved in the direction $-\mathbf{c} = (1, 3)$ as much as possible until the optimal point $(4/3, 14/3)$ is reached.

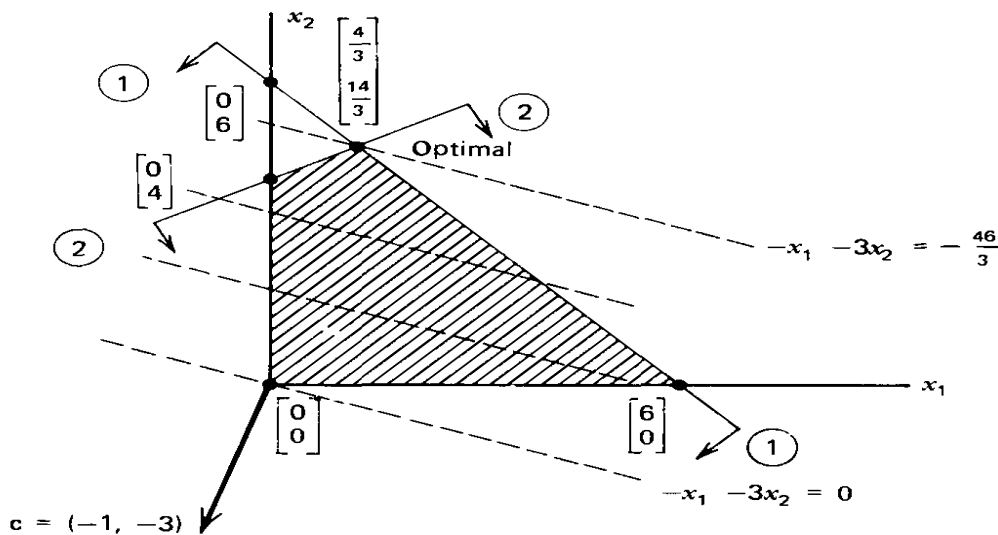


Figure 3.6 Numerical example

In this example we had a unique optimal solution. Other cases may occur depending upon the problem structure. All possible cases that may arise are summarized below (for a minimization problem).

1. *Unique Finite Optimal Solution.* If the optimal finite solution is unique, then it occurs at an extreme point. Figures 3.7a and b show a unique

optimal solution. In Figure 3.7a the feasible region is bounded; that is, there is a ball around the origin that contains the feasible region. In Figure 3.7b the feasible region is not bounded. In each case, however, the unique optimal solution is finite.

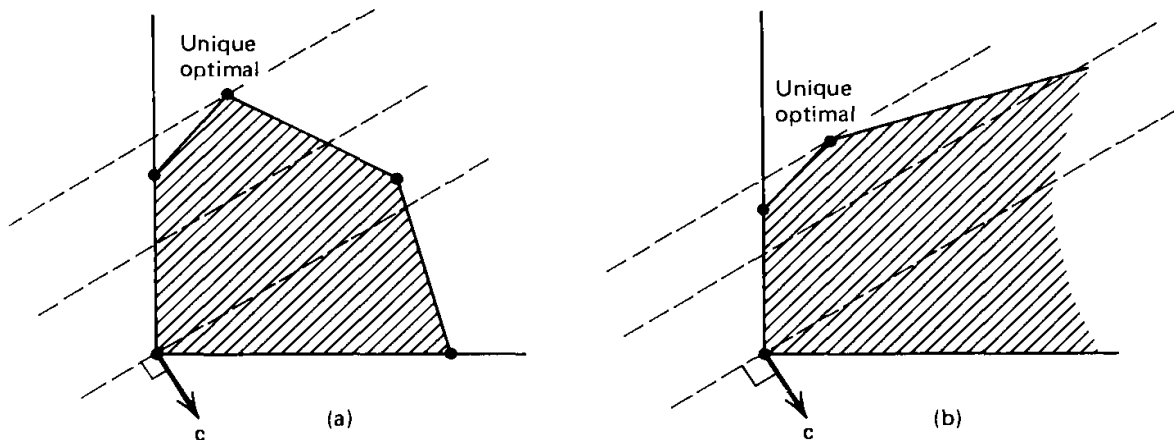


Figure 3.7. **Unique finite optimal solution: (a) Bounded region. (b) Unbounded Region.**

2. *Alternative Finite Optimal Solutions.* This case is illustrated in Figure 3.8. Note that in Figure 3.8a the feasible region is bounded. The two corner points x_1^* and x_2^* are optimal, and also any point on the line segment joining them. In Figure 3.8b the feasible region is unbounded but the optimal objective is finite. Any point on the “ray” with vertex x^* in Figure 3.8b is optimal.

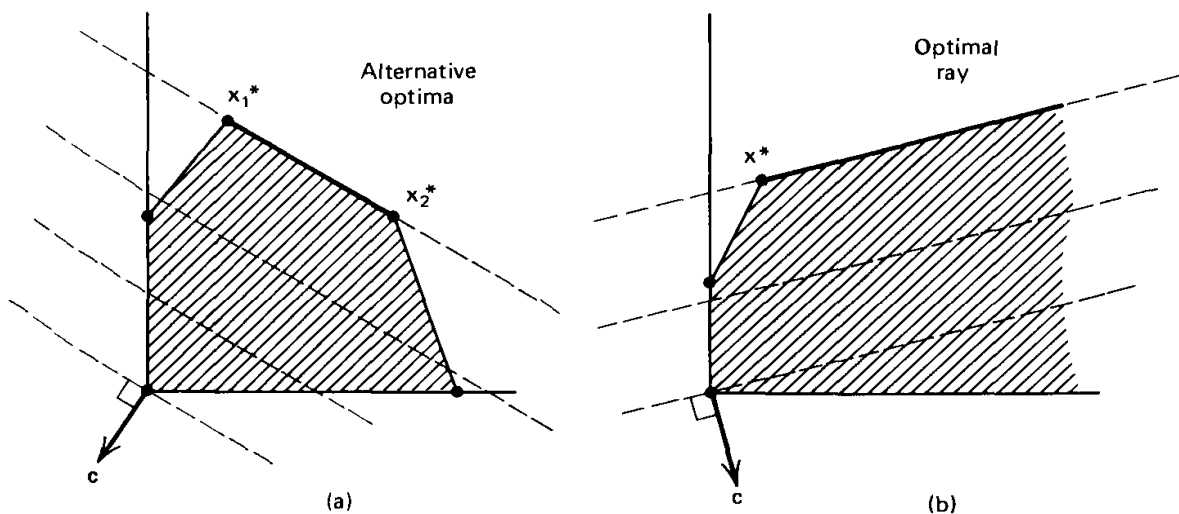


Figure 3.8. **Alternative finite optima: (a) Bounded Region. (b) Unbounded Region.**

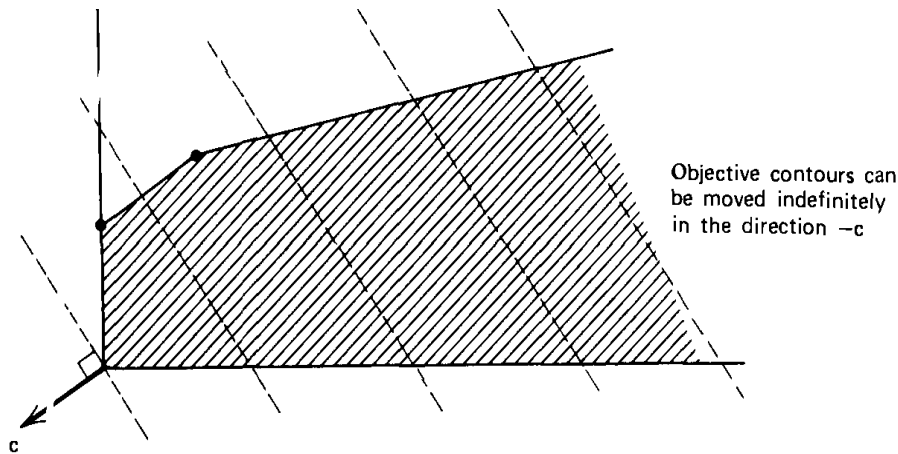


Figure 3.9. **Unbounded optimal solution.**

3. *Unbounded Optimal Solution.* This case is illustrated in Figure 3.9 where both the feasible region and the optimal solution are unbounded. For a minimization problem the plane $\mathbf{c}\mathbf{x} = z$ can be moved in the direction $-\mathbf{c}$ indefinitely while always intersecting with the feasible region. In this case the optimal objective is unbounded with value $-\infty$.
4. *Empty Feasible Region.* In this case the system of equations and/or inequalities defining the feasible region is *inconsistent*. To illustrate, consider the following problem.

$$\text{Minimize } -2x_1 + 3x_2$$

$$\text{Subject to } -x_1 + 2x_2 \leq 2$$

$$2x_1 - x_2 \leq 3$$

$$x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

Examining Figure 3.10, it is clear that there exists no point (x_1, x_2) satisfying the above inequalities. The problem is said to be *infeasible*, *inconsistent*, or with *empty feasible region*.

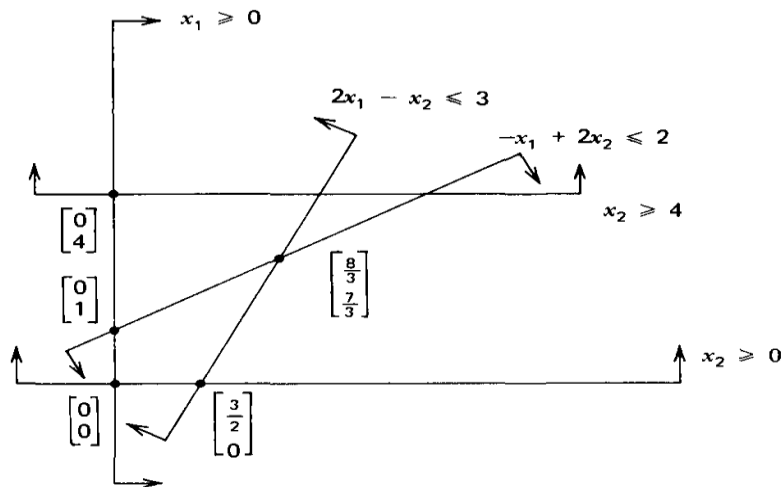


Figure 3.10. An example of an empty feasible region.

Theorem: (Existence of optimal solution)

- i) If S is bounded feasible region, then the LPP has max. and min. value of S .
- ii) If S is unbounded feasible set, then max or min value of an LPP may not exist. But, if S is unbounded and the coefficients of z are positive, then z has min. value but has no max. value.
- iii) If S is empty set, then the LPP is infeasible.

Exercise: Solve the following LPP using graphical method (Corner point and Iso-profit (Iso-cost) methods).

- max $z = 8x + 10y$
 s. t $2x + y \leq 50$
 $x + 2y \leq 70$
 $x, y \geq 0$

Ans: The optimal solution is (10,30) and the value of objective function is 380.

- max $z = 20x + 30y$
 s. t $x + y \geq 50$
 $x \leq 30$
 $y \leq 40$
 $x, y \geq 0$

Ans: The optimal solution is (30,20) and the value of objective function is 1200.

- max $z = 30x + 40y$
 s. t $2x + y \geq 12$
 $x + y \geq 9$
 $x + 3y \geq 15$
 $x, y \geq 0$

Ans: The optimal solution is (6,3) and the value of objective function is 300.

- max $z = x + y$
 s. t $x + 2y \leq 10$
 $3x + 2y \leq 18$
 $x, y \geq 0$

Ans: The optimal solution is (4,3) and the value of objective function is 7.

2.4. THE SIMPLEX METHOD

The simplex method computations are particularly tedious, repetitive, and, above all, boring. As you do these computations, you should not lose track of the big picture; namely, the simplex method attempts to move from one corner point of the solution space to a better corner point until the optimum is found. In this method the LPP must be expressed in the standard form. Simplex method is also called algebraic method.

The development of the simplex method computations is facilitated by imposing two requirements on the constraints of the problem:

1. All the constraints (with the exception of the non-negativity of the variables) are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

These two requirements are imposed here primarily to standardize and streamline the simplex method calculations. It is important to know that all commercial packages directly accept inequality constraints, nonnegative right-hand side, and unrestricted variables. Any necessary preconditioning of the model is done internally in the software before the simplex method solves the problem.

2.4.1. Linear programs in standard form

Any LPP problem can be converted to the standard minimization form. To write any LPP in the standard minimization form:

- ✓ Express the objective function as minimization problem.
- ✓ Transform all constraints to equality constraints by introducing nonnegative variables called slack or surplus variables with nonnegative right hand side.
- ✓ Handle unrestricted and non-positive variables.

Note:

- ❖ Constant term in the objective function will not change the set of optimal solutions and hence we can remove the constant term.
 $\max(z + p) = \max z + \max p = \max z + p$, for some constant p .
 $\min(z + p) = \min z + \min p = \min z + p$, for some constant p .
- ❖ $\max z = < c, x >$ can be replaced by $\min \bar{z} = -< c, x >$ and we have $\max z = -\min \bar{z}$.
- ❖ A linear constraint that involves " $\leq$ " can be converted in to an equality by adding a new nonnegative variable to the left hand side of the inequality. Such a variable is numerically equal to the difference between the right and left hand sides of the inequality and is known as a slack variable. It represents the waste involved in that phase of the system modeled by the constraint.

- ❖ A linear constraint that involves " $\geq$ " can be converted in to an equality by subtracting a new nonnegative variable from the left hand side of the inequality. Such a variable is numerically equal to the difference between the left and right hand sides of the inequality and is known as a surplus variable. It represents excess input into that phase of the system modeled by the constraint.
- ❖ If $b_i < 0$ in the constraint set of the LPP, then multiply the i^{th} constraint by -1.
- ❖ **Unrestricted and non-positive variables**
 - If we have a non-positive variable $x_j \leq 0$, then by substituting $x_j' := -x_j \geq 0$ will be satisfy the non-negativity condition.
 - If we have unrestricted variable $x_j \in \mathbb{R}$, then we substitute $x_j = x_j' - x_j''$; $x_j', x_j'' \geq 0$.

Example:

1. Transform the following problems to the standard form of LPP

$$\begin{array}{ll}
 \text{a)} & \max \quad z = 2x_1 - x_2 + 2x_3 - x_4 \\
 & \text{s.t. } 3x_1 + x_2 + 2x_3 + 4x_4 \leq 5 \\
 & \quad 6x_1 - x_2 - x_3 - x_4 \geq 3 \\
 & \quad -7x_1 + 5x_2 + x_3 + 6x_4 = -6 \\
 & \quad x_1, x_2 \geq 0, \quad x_3 \in \mathbb{R}, \quad x_4 \leq 0 \\
 \text{b)} & \min \quad z = 2x_1 + x_2 - 6x_3 - x_4 \\
 & \text{s.t. } 3x_1 + x_2 + 2x_3 + 4x_4 \leq 5 \\
 & \quad 2x_1 + 6x_2 + x_4 \leq 25 \\
 & \quad 10 \geq 4x_1 + 5x_2 \geq -6 \\
 & \quad x_1 + x_2 + 2x_3 + 2x_4 = -20 \\
 & \quad x_1, x_2 \geq 0, \quad x_4, x_3 \in \mathbb{R}
 \end{array}$$

- 2.

Consider the following linear programming problem.

$$\text{Minimize} \quad x_1 - 2x_2 - 3x_3$$

$$\text{Subject to} \quad x_1 + x_2 + x_3 \leq 6$$

$$x_1 + 2x_2 + 4x_3 \geq 12$$

$$x_1 - x_2 + x_3 \geq 2$$

$$x_1, x_2, x_3 \text{ unrestricted}$$

Convert the problem in to standard maximization problem.

- 3.

Sketch the feasible region of the set $\{x : Ax \leq b\}$ where A and b are given below. In each case state whether the feasible region is empty or not, and whether it is bounded or not.

$$\text{a. } A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 6 \\ 2 \end{bmatrix}$$

$$\text{b. } A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 5 \end{bmatrix}$$

$$\text{c. } A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \\ -1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ -12 \\ 0 \end{bmatrix}$$

4.

Consider the following problem.

$$\text{Maximize } 2x_1 + 3x_2$$

$$\text{Subject to } x_1 + x_2 \leq 2$$

$$4x_1 + 6x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

- Sketch the feasible region.
- Find two alternative optimal extreme (corner) points.
- Find an infinite class of optimal solutions.

2.4.2. Basic feasible solutions (BFS)

Consider the system $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an m vector. Suppose that $\text{rank}(\mathbf{A}, \mathbf{b}) = \text{rank}(\mathbf{A}) = m$. After possibly rearranging the columns of $\mathbf{A}$, let $\mathbf{A} = [\mathbf{B}, \mathbf{N}]$ where $\mathbf{B}$ is an $m \times m$ invertible matrix and $\mathbf{N}$ is an $m \times (n - m)$ matrix. The point $\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$ where

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_N = \mathbf{0}$$

is called a *basic solution* of the system. If $\mathbf{x}_B \geq \mathbf{0}$, then $\mathbf{x}$ is called a *basic feasible*

solution of the system. Here $\mathbf{B}$ is called the *basic matrix* (or simply the basis) and $\mathbf{N}$ is called the *nonbasic matrix*. The components of $\mathbf{x}_B$ are called *basic variables*, and the components of $\mathbf{x}_N$ are called *nonbasic variables*. If $\mathbf{x}_B > \mathbf{0}$, then $\mathbf{x}$ is called a *nondegenerate basic feasible solution*, and if at least one component of $\mathbf{x}_B$ is zero, then $\mathbf{x}$ is called a *degenerate basic feasible solution*.

Each **basic solution** has m **basic variables**, and the rest of the variables are **non-basic variables** set equal to zero. (The number of non-basic variables equals n plus the number of surplus variables.) The values of the **basic variables** are given by the simultaneous solution of the system of m equations for the problem in augmented form (after the non-basic variables are set to zero). This basic solution is the augmented corner-point solution whose n defining equations are those indicated by the non-basic variables.

Now consider the *basic feasible* solutions. Note that the only requirements for a solution to be feasible in the augmented form of the problem are that it satisfies the system of equations and that *all* the variables be *nonnegative*.

A **basic feasible solution (BFS)** is a basic solution where all m basic variables are nonnegative (≥ 0). A basic solution (BF) solution is said to be **degenerate** if any of these m variables equals zero.

Thus, it is possible for a variable to be zero and still not be a non-basic variable for the current BF solution. Therefore, it is necessary to keep track of which is the current set of non-basic variables (or the current set of basic variables) rather than to rely upon their zero values.

Remark: We choose linearly independent vectors from n vectors, so that the maximum number of basic matrix is $\binom{n}{m} = \frac{n!}{(n-m)!m!}$ and hence the maximum number of basic feasible solution is $\binom{n}{m}$.

Example 1. Consider the polyhedral set defined by the following inequalities:

$$x_1 + x_2 \leq 6$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

By introducing the slack variables x_3 and x_4 , the problem is put in the following standard format:

$$x_1 + x_2 + x_3 = 6$$

$$x_2 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Note that the constraint matrix $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. From the

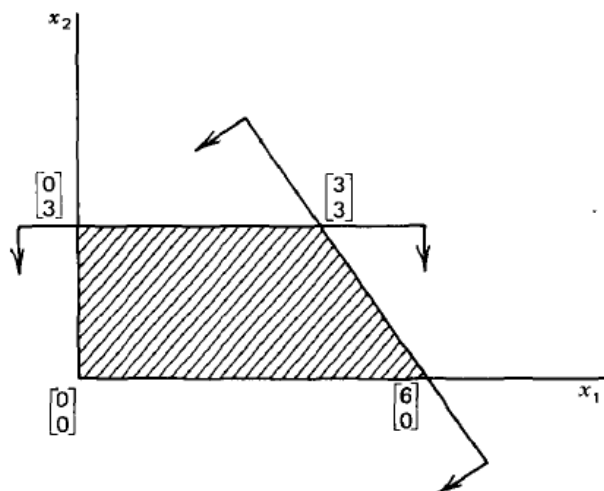


Figure 3.4.1 Basic feasible solution

foregoing definition, basic feasible solutions correspond to finding a 2×2 basis $\mathbf{B}$ with nonnegative $\mathbf{B}^{-1}\mathbf{b}$. The following are the possible ways of extracting $\mathbf{B}$ out of $\mathbf{A}$.

$$1. \mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2. \mathbf{B} = [\mathbf{a}_1, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3. \mathbf{B} = [\mathbf{a}_2, \mathbf{a}_3] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4. \mathbf{B} = [\mathbf{a}_2, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5. \mathbf{B} = [\mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that the points corresponding to 1, 2, 3, and 5 above are basic feasible solutions. The point obtained in 4 is a basic solution, but is not feasible because it violates the nonnegativity restrictions. In other words, we have four basic feasible solutions, namely

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 3 \end{bmatrix}$$

These points belong to E^4 since after introducing the slack variables we have four variables. These basic feasible solutions, projected in E^2 —that is, in the (x_1, x_2) space—give rise to the following four points:

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In this example, the possible number of basic feasible solutions is bounded by the number of ways of extracting two columns out of four columns to form the basis. Therefore the number of basic feasible solutions is less or equal to

$$\binom{4}{2} = \frac{4!}{2!2!} = 6.$$

Out of these six possibilities, one point violates the non-negativity of $\mathbf{B}^{-1}\mathbf{b}$. Furthermore, a_1 and a_2 have could not have been used to form a basis since $a_1 = a_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are linearly dependent, and hence the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ does not qualify as a basis.

Example 2.

Consider the following system of inequalities:

$$x_1 + x_2 \leq 6$$

$$x_2 \leq 3$$

$$x_1 + 2x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

Note that the feasible region is precisely the region of example 1 above, since the third restriction $x_1 + 2x_2 \leq 9$ is “redundant”. After adding the slack variables x_3, x_4 and x_5 , we get

$$x_1 + x_2 + x_3 = 6$$

$$x_2 + x_4 = 3$$

$$x_1 + 2x_2 + x_5 = 9$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

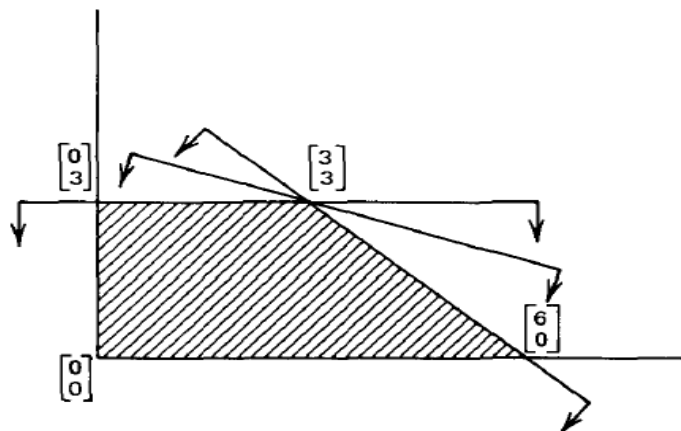


Figure 3.4.2 Degenerate basic feasible solution

Note that

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

Let us consider the basic feasible solution with $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$.

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that this basic feasible solution is degenerate since the basic variable $x_3 = 0$. Now consider the basic feasible solution with $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4]$.

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that this basic feasible solution gives rise to the same point obtained by $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$. It can be also checked that the basic feasible solution with basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_5]$ is given by

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note that all three of the foregoing basic feasible solutions with different *bases* are represented by the single extreme point $(x_1, x_2, x_3, x_4, x_5) = (3, 3, 0, 0, 0)$. Each of the three basic feasible solutions is degenerate since each contains a basic variable at level zero. The remaining extreme points of Figure 3.4.2 correspond to nondegenerate basic feasible solutions (why?).

Theorem : All basic feasible solution of $Ax = b$, $b \geq 0$ are extreme points of the feasible region. That is, a point x is an extreme point of the set $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ if and only if x is basic feasible solution.

Theorem: (Fundamental Theorem of LPP)

$$z = Cx$$

Consider the linear programming problem(LPP) $\min s.t \ Ax = b$, where A is m by n matrix with $x \geq 0$

$rank(A) = m$. If the LPP has a finite optimal solution, then the optimal solution is attained at a basic faesible solution(BFS).

Since the number of basic feasible solutions is bounded by $\binom{n}{m}$, one may think of simply listing all basic feasible solutions, and picking the one with the minimal objective value. This is not satisfactory, however, for a number of reasons. Firstly, the number of basic feasible solutions is bounded by $\binom{n}{m}$, which is large, even for moderate values of m and n . Secondly, this simple approach does not tell us if the problem has an unbounded solution that may occur if the feasible region is unbounded. Lastly, if the feasible region is empty, and if we apply the foregoing “simple-minded procedure,” we shall discover that the feasible region is empty, only after all possible ways of extracting m columns out of n columns of the matrix A fail to produce a basic feasible solution, on the grounds that B does not have an inverse, or else $B^{-1}b \not\geq 0$.

The simplex method is a clever procedure that moves from an extreme point to another extreme point, with a better (at least not worse) objective. It also discovers whether the feasible region is empty and whether the optimal solution is unbounded. In practice, the method only enumerates a small portion of the extreme points of the feasible region.

2.4.3. Improving a basic feasible solution

Given a basic feasible solution, we shall describe a method for obtaining a new basic feasible solution with a better objective value. This is the foundation of the simplex method.

Consider the following linear programming problem.

$$\text{Minimize } cx$$

$$\text{Subject to } Ax = b$$

$$x \geq 0$$

where A is an $m \times n$ matrix with rank m . Suppose that we have a basic feasible solution $\begin{pmatrix} B^{-1}b \\ 0 \end{pmatrix}$ whose objective value z_0 is given by

$$z_0 = \mathbf{c} \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} = (\mathbf{c}_B, \mathbf{c}_N) \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix} = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b} \quad (3.1)$$

Now let $\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix}$ be an arbitrary feasible solution. Then $\mathbf{x}_B \geq \mathbf{0}$, $\mathbf{x}_N \geq \mathbf{0}$, and $\mathbf{b} = \mathbf{Ax} = \mathbf{Bx}_B + \mathbf{Nx}_N$. Multiplying by $\mathbf{B}^{-1}$ and rearranging the terms, we get

$$\begin{aligned} \mathbf{x}_B &= \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Nx}_N \\ &= \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in R} \mathbf{B}^{-1}\mathbf{a}_j x_j \end{aligned} \quad (3.2)$$

where R is the current set of the indices of the nonbasic variables. Noting Equations (3.2) and (3.1), and letting z denote the objective function at $\mathbf{x}$, we get

$$\begin{aligned} z &= \mathbf{cx} \\ &= \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N \\ &= \mathbf{c}_B \left(\mathbf{B}^{-1}\mathbf{b} - \sum_{j \in R} \mathbf{B}^{-1}\mathbf{a}_j x_j \right) + \sum_{j \in R} c_j x_j \\ &= z_0 - \sum_{j \in R} (z_j - c_j) x_j \end{aligned} \quad (3.3)$$

where $z_j = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_j$ for each nonbasic variable.

Equation (3.3) can guide us in the process of improving the current basic feasible solution. Since we are to minimize z , it would be to our advantage to increase x_j (from its current level of zero) whenever $z_j - c_j > 0$. The following rule will be adopted. Fix each nonbasic variable x_j at zero, except for one nonbasic variable x_k with a positive $z_k - c_k$ ($z_k - c_k$ is the most positive $z_j - c_j$, say). From Equation (3.3), the new objective value z is given by

$$z = z_0 - (z_k - c_k)x_k \quad (3.4)$$

Since $z_k - c_k > 0$, it would be to our benefit to increase x_k as much as possible. As x_k is increased, the current basic variables must be modified according to Equation (3.2), and hence $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_k x_k = \bar{\mathbf{b}} - \mathbf{y}_k x_k$, where $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$ and $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b}$. Denoting the components of $\mathbf{x}_B$ and $\bar{\mathbf{b}}$ by $x_{B_1}, x_{B_2}, \dots, x_{B_m}$ and $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_m$, the preceding vector equation reads as follows:

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_r} \\ \vdots \\ x_{B_m} \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_r \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ y_{2k} \\ \vdots \\ y_{rk} \\ \vdots \\ y_{mk} \end{bmatrix} x_k \quad (3.5)$$

If $y_{ik} \leq 0$, then x_{B_i} increases as x_k increases and so x_{B_i} continues to be nonnegative. If $y_{ik} > 0$, then x_{B_i} will decrease as x_k increases. In order to satisfy nonnegativity, x_k is increased until the first point at which a basic variable x_{B_r} drops to zero. Examining Equation (3.5), it is then clear that the first basic variable dropping to zero corresponds to the minimum of $\bar{b}_i/y_{ik}$ for positive y_{ik} . More precisely,

$$\frac{\bar{b}_r}{y_{rk}} = \text{Minimum}_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\} = x_k \quad (3.6)$$

In the absence of degeneracy $\bar{b}_r > 0$, and hence $x_k = \bar{b}_r/y_{rk} > 0$. From Equation (3.4) and the fact that $z_k - c_k > 0$, it then follows that $z < z_0$, and the objective function strictly improves. As x_k increases from level 0 to $\bar{b}_r/y_{rk}$, a new feasible solution is obtained. Substituting $x_k = \bar{b}_r/y_{rk}$ in Equation (3.5) gives the

following point:

$$x_{B_i} = \bar{b}_i - \frac{y_{ik}}{y_{rk}} \bar{b}_r \quad i = 1, 2, \dots, m$$

$$x_k = \frac{\bar{b}_r}{y_{rk}} \quad (3.7)$$

all other x_j 's are zero

From Equation (3.7) above, $x_{B_r} = 0$ and hence at most m variables are positive. The corresponding columns are $\mathbf{a}_{B_1}, \mathbf{a}_{B_2}, \dots, \mathbf{a}_{B_{r-1}}, \mathbf{a}_k, \mathbf{a}_{B_{r+1}}, \dots, \mathbf{a}_{B_m}$. Note that these columns are linearly independent since $y_{rk} \neq 0$. (Recall that if $\mathbf{a}_{B_1}, \dots, \mathbf{a}_{B_r}, \dots, \mathbf{a}_{B_m}$ are linearly independent, and if $\mathbf{a}_k$ replaces $\mathbf{a}_{B_r}$, then the new columns are linearly independent if and only if $y_{rk} \neq 0$; see Section 2.1). Therefore the point given by Equation (3.7) is a basic feasible solution.

To summarize, we have described a procedure that moves from a basic feasible solution to another basic feasible solution. This is done by increasing the value of a nonbasic variable x_k with positive $z_k - c_k$ and adjusting the current basic variables. In the process, the variable x_{B_r} drops to zero. The variable x_k is said to enter the basis and x_{B_r} is said to leave the basis. In the absence of degeneracy the objective function value strictly decreases and hence the generated points are distinct. Since there is only a finite number of basic feasible solutions, the procedure would terminate in a finite number of steps.

Example 3.

$$\text{Minimize } x_1 + x_2$$

$$\text{Subject to } x_1 + 2x_2 \leq 4$$

$$x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Introduce the slack variables x_3 and x_4 to put the problem in a standard form. This leads to the following constraint matrix $\mathbf{A}$:

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Consider the basic feasible solution corresponding to $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2]$. In other words, x_1 and x_2 are the basic variables while x_3 and x_4 are the nonbasic variables.

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This point is shown in Figure 3.5. In order to improve this basic feasible solution, calculate $z_j - c_j$ for the nonbasic variables.

$$\begin{aligned}
 z_3 - c_3 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_3 - c_3 \\
 &= (1, 1) \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 \\
 &= (1, 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 z_4 - c_4 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_4 - c_4 \\
 &= (1, 1) \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 \\
 &= (1, 1) \begin{pmatrix} -2 \\ 1 \end{pmatrix} - 0 \\
 &= -1
 \end{aligned}$$

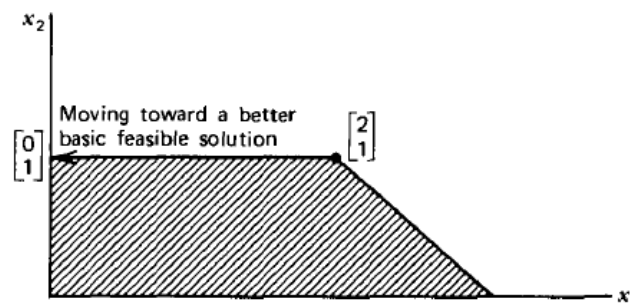


Figure 3.5. Improving a basic feasible solution.

Since $z_3 - c_3 > 0$, then the objective improves by increasing x_3 . The modified solution is given by

$$\begin{aligned}
 \mathbf{x}_B &= \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{a}_3 x_3 \\
 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3
 \end{aligned}$$

The maximum value of x_3 is 2 (any larger value of x_3 will force x_1 to be negative). Therefore the new basic feasible solution is

$$(x_1, x_2, x_3, x_4) = (0, 1, 2, 0)$$

Here x_3 enters the basis and x_1 leaves the basis. Note that the new point has an objective value equal to 1, which is an improvement over the previous objective value of 3. The improvement is precisely $(z_3 - c_3)x_3 = 2$. The reader is asked to continue the improvement process starting from the new point.

Interpretation of Entering and Leaving the Basis

We now look more closely at the process of entering and leaving the basis, and their interpretation.

INTERPRETATION OF $z_k - c_k$

The criterion $z_k - c_k > 0$ for a nonbasic variable x_k to enter the basis, will be used over and over again throughout the text. It will be helpful at this stage to review the definition of z_k , and make a few comments on the meaning of the entry criterion $z_k - c_k > 0$. Recall that $z = \mathbf{c}_B \mathbf{b} - (z_k - c_k)x_k$, where

$$z_k = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_k = \mathbf{c}_B \mathbf{y}_k = \sum_{i=1}^m c_{B_i} y_{ik} \quad (3.8)$$

and c_{B_i} is the cost of the i th basic variable. Note that if x_k is raised from zero level, while the other nonbasic variables are kept at zero level, then the basic variables $x_{B_1}, x_{B_2}, \dots, x_{B_m}$ must be modified according to Equation (3.5). In other words, if x_k is increased by 1 unit, then $x_{B_1}, x_{B_2}, \dots$, and x_{B_m} will be decreased respectively by $y_{1k}, y_{2k}, \dots, y_{mk}$ units (if $y_{ik} < 0$, then x_{B_i} will be increased). The saving (a negative saving means more cost) that results from the modification of the basic variables, as a result of increasing x_k by 1 unit, is therefore, $\sum_{i=1}^m c_{B_i} y_{ik}$, which is z_k (see Equation 3.8). However, the cost of increasing x_k itself by 1 unit is c_k . Hence $z_k - c_k$ is the saving minus the cost

of increasing x_k by 1 unit. Naturally, if $z_k - c_k$ is positive, it will be to our advantage to increase x_k . For each unit of x_k , the cost will be reduced by an amount $z_k - c_k$ and hence it will be to our advantage to increase x_k as much as possible. On the other hand, if $z_k - c_k < 0$, then by increasing x_k , the net saving is negative, and this action will result in a larger cost. So this action is

prohibited. Finally if $z_k - c_k = 0$, then increasing x_k will lead to a different solution, with the same cost. So whether x_k is kept at zero level, or increased, no change in cost takes place.

Now suppose that x_k is a basic variable. In particular, suppose that x_k is the t th basic variable, that is, $x_k = x_{B_t}$, $c_k = c_{B_t}$, and $\mathbf{a}_k = \mathbf{a}_{B_t}$. Recall that $z_k = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_k = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_{B_t}$. But $\mathbf{B}^{-1} \mathbf{a}_{B_t}$ is a vector of zeros except for one at the t th position (see Exercise 2.12). Therefore, $z_k = c_{B_t}$, and hence $z_k - c_k = c_{B_t} - c_{B_t} = 0$.

Leaving the Basis and the Blocking Variable

Suppose that we decided to increase a nonbasic variable x_k with a positive $z_k - c_k$. From Equation (3.4), the larger the value of x_k , the smaller is the objective z . As x_k is increased, the basic variables are modified according to Equation (3.5). If the vector $\mathbf{y}_k$ has any positive component(s), then the corresponding basic variable(s) is decreased as x_k is increased. Therefore the nonbasic variable x_k cannot be indefinitely increased, because otherwise the nonnegativity of the basic variables will be violated. The first basic variable x_{B_t} that drops to zero is called the *blocking variable* because it blocked further increase of x_k .

Thus x_k enters the basis and x_{B_t} leaves the basis.

Example 4.

$$\text{Minimize } 2x_1 - x_2$$

$$\text{Subject to } -x_1 + x_2 \leq 2$$

$$2x_1 + x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Introduce the slack variables x_3 and x_4 . This leads to the following constraints:

$$-x_1 + x_2 + x_3 = 2$$

$$2x_1 + x_2 + x_4 = 6$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Consider the basic feasible solution with basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ and

$$\mathbf{B}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N$$

$$\begin{aligned} \begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{3} \\ \frac{10}{3} \end{bmatrix} - \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} x_3 - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} x_4 \end{aligned} \quad (3.9)$$

Currently $x_3 = x_4 = 0$, $x_1 = \frac{4}{3}$ and $x_2 = \frac{10}{3}$. Note that

$$z_4 - c_4 = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_4 - c_4 = (2, -1) \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = \frac{1}{3} > 0$$

Hence the objective improves by introducing x_4 in the basis. Then x_3 is kept at zero level, x_4 is increased, and x_1 and x_2 are modified according to Equation (3.9). We see that x_4 can be increased to 4, at which instant x_1 drops to zero. Any further increase of x_4 results in violating the nonnegativity of x_1 , and so x_1 is called the *blocking variable*. With $x_4 = 4$ and $x_3 = 0$, the modified values of x_1 and x_2 are 0 and 2 respectively. The new basic feasible solution is

$$(x_1, x_2, x_3, x_4) = (0, 2, 0, 4)$$

Note that $\mathbf{a}_4$ replaces $\mathbf{a}_1$; that is, x_1 drops from the basis and x_4 enters the basis. The new set of basic and nonbasic variables are given below:

$$\mathbf{x}_B = \begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Moving from the old to the new basic feasible solution is illustrated in Figure 3.6. Note that as x_4 increases by 1 unit, x_1 decreases by $\frac{1}{3}$ unit and x_2 decreases by $\frac{1}{3}$ unit; that is, we move in the direction $(-\frac{1}{3}, -\frac{1}{3})$ in the (x_1, x_2) space. This continues until we are blocked by the nonnegativity restriction $x_1 \geq 0$. At this point x_1 drops to zero and leaves the basis.

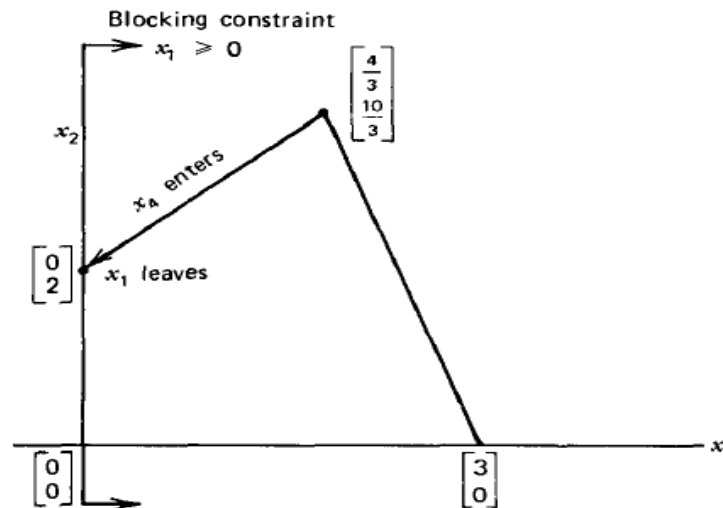


Figure 3.6. **Blocking variable (Constraint).**

2.4.4. Termination: Optimality and Unboundedness

We have discussed a procedure that moves from one basic feasible solution to an improved basic feasible solution, by introducing one variable into the basis, and removing another variable from the basis. The criteria for entering and leaving are summarized below.

1. *Entering:* x_k may enter if $z_k - c_k > 0$
2. *Leaving:* x_{B_r} may leave if

$$\frac{\bar{b}_r}{y_{rk}} = \text{Minimum}_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

Two logical questions immediately arise. What happens if each nonbasic variable x_j has $z_j - c_j \leq 0$? In this case no nonbasic variable is eligible for entering the basis. Second, suppose that $z_k - c_k > 0$, and hence x_k is eligible to enter the basis, but we cannot find any positive component y_{ik} , that is, $y_k \leq 0$. As the reader may have suspected, the first case says that we have already reached the optimal solution, and the second case says that the optimal solution is unbounded. These cases will be discussed in more detail in this section.

Termination with an Optimal Solution

Consider the following problem, where A is an $m \times n$ matrix with rank m .

Minimize cx

Subject to $Ax = b$

$x \geq 0$

Suppose that $\mathbf{x}^*$ is a basic feasible solution with basis $\mathbf{B}$; that is, $\mathbf{x}^* = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{pmatrix}$.

Let z^* denote the objective of $\mathbf{x}^*$, that is, $z^* = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b}$. Suppose further that $z_j - c_j \leq 0$ for all nonbasic variables, and hence there are no nonbasic variables that are eligible to enter the basis. Let $\mathbf{x}$ be any feasible solution with objective value z . Then from Equation (3.3) we have

$$z^* - z = \sum_{j \in R} (z_j - c_j) x_j \quad (3.10)$$

Since $z_j - c_j \leq 0$ and $x_j \geq 0$ for all variables, then $z^* \leq z$. This holds for every feasible vector $\mathbf{x}$ and therefore $\mathbf{x}^*$ is an optimal basic feasible solution.

Unique and Alternative Optimal Solutions

We can get more information from Equation (3.10). If $z_j - c_j < 0$ for all nonbasic components, then the current optimal solution is unique. To show this, let $\mathbf{x}$ be any feasible solution that is distinct from $\mathbf{x}^*$. Then there is at least one nonbasic component x_j that is positive, because if all nonbasic components are zero, $\mathbf{x}$ would not be distinct from $\mathbf{x}^*$. From Equation (3.10) it follows that $z > z^*$, and hence $\mathbf{x}^*$ is the unique optimal solution.

Now consider the case where $z_j - c_j \leq 0$ for all nonbasic components, but $z_k - c_k = 0$ for at least one nonbasic variable x_k . As x_k is increased, we get (in the absence of degeneracy) points that are distinct from $\mathbf{x}^*$ but have the same objective value (why?). If x_k is increased until it is blocked by a basic variable, we get an alternative optimal basic feasible solution. The process of increasing x_k from level zero until it is blocked generates an infinite number of alternative optimal solutions.

Example 5.

$$\text{Minimize } -3x_1 + x_2$$

$$\text{Subject to } x_1 + 2x_2 + x_3 = 4$$

$$-x_1 + x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Consider the basic feasible solution with basis $\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. The corresponding point is given by

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the objective value is -12 . To see if we can improve this solution, calculate $z_2 - c_2$ and $z_3 - c_3$:

$$\begin{aligned} z_2 - c_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\ &= (-3, 0) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \\ &= (-3, 0) \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 1 \\ &= -7 \end{aligned}$$

$$\begin{aligned} z_3 - c_3 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_3 - c_3 \\ &= (-3, 0) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 \\ &= (-3, 0) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0 \\ &= -3 \end{aligned}$$

Since both $z_2 - c_2 < 0$ and $z_3 - c_3 < 0$, then the basic feasible solution $(x_1, x_2, x_3, x_4) = (4, 0, 0, 5)$ is the unique optimal point. This unique optimal solution is illustrated in Figure 3.7a. Now consider a new problem where the objective function $-2x_1 - 4x_2$ is to be minimized over the same region. Again, consider the same point $(4, 0, 0, 5)$. The objective value is -8 . Calculate $z_2 - c_2$

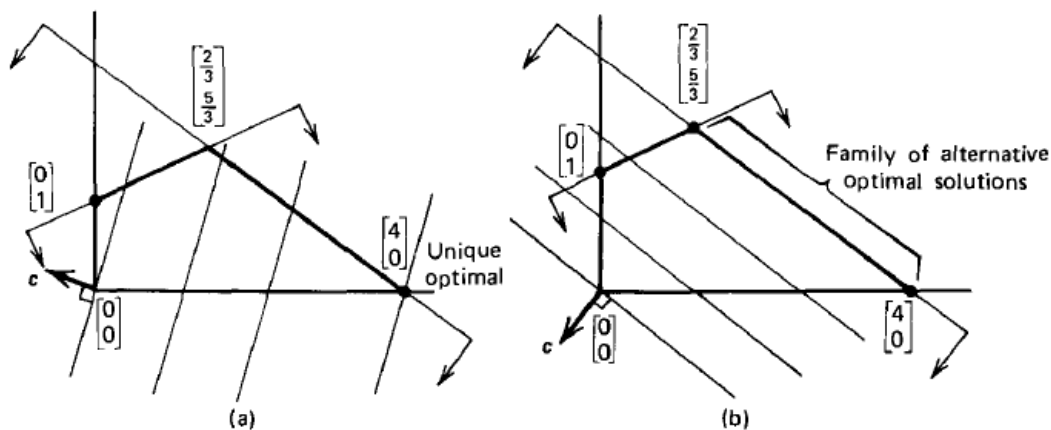


Figure 3.7. Termination criterion: (a) Unique optimal. (b) Alternative optima.

and $z_3 - c_3$ as follows:

$$\begin{aligned} z_2 - c_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\ &= (-2, 0) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} z_3 - c_3 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_3 - c_3 \\ &= (-2, 0) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0 \\ &= -2 \end{aligned}$$

In this case, the given basic feasible solution is optimal, but it is no longer a unique optimal solution. We see that by increasing x_2 , a class of optimal solutions is obtained. Actually, if we increase x_2 , keep $x_3 = 0$, and modify x_1 and x_4 , we get

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} &= \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{a}_2 x_2 \\ &= \begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} x_2 \end{aligned}$$

For any $x_2 \leq \frac{5}{3}$, the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 2x_2 \\ x_2 \\ 0 \\ 5 - 3x_2 \end{bmatrix}$$

is an optimal solution with objective -8 . In particular, if $x_2 = \frac{5}{3}$, we get an alternative basic feasible optimal solution, where x_4 drops from the basis. This is illustrated in Figure 3.7b. Note that the new objective function lines are parallel to the hyperplane $x_1 + x_2 = 4$ corresponding to the first constraint. That is why we obtain alternative optimal solutions.

Unboundedness

Suppose that we have a basic feasible solution of the system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, with objective value z_0 . Now let us consider the case when we find a nonbasic variable x_k with $z_k - c_k > 0$ and $\mathbf{y}_k \leq \mathbf{0}$. This variable is eligible to enter the

basis since increasing it will improve the objective function. From Equation (3.3) we have

$$z = z_0 - (z_k - c_k)x_k$$

Since we are minimizing the objective z , and since $z_k - c_k > 0$, then it is to our benefit to increase x_k indefinitely, which will make z go to $-\infty$. The reason that we were not able to do this before was that the increase in the value of x_k was blocked by a basic variable. This puts a “ceiling” on x_k beyond which a basic variable will be negative. But if blocking is not encountered, there is no reason why we should stop increasing x_k . This is precisely the case when $y_k \leq 0$. Recall that from Equation (3.5) we have

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{y}_k x_k$$

and so if $y_k \leq 0$, then x_k can be increased indefinitely without any of the basic variables becoming negative. Therefore the solution $\mathbf{x}$ (where $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{y}_k x_k$, x_k is arbitrarily large and other nonbasic components are zero) is feasible and its objective value $z = z_0 - (z_k - c_k)x_k$, which approaches $-\infty$ as x_k approaches $+\infty$.

To summarize, if we have a basic feasible solution with $z_k - c_k > 0$ for some nonbasic variable x_k , and meanwhile $y_k \leq 0$, then the optimal is unbounded with objective $-\infty$. This is obtained by increasing x_k indefinitely and adjusting the values of the current basic variables, and is equivalent to moving along the ray:

$$\left\{ \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_k \begin{bmatrix} -y_k \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} : x_k \geq 0 \right\}$$

Note that the vertex of the ray is the current basic feasible solution $\begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{pmatrix}$ and the direction of the ray is

$$\mathbf{d} = \begin{bmatrix} -y_k \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where the 1 appears in the k th position. It may be noted that

$$\mathbf{cd} = (\mathbf{c}_B, \mathbf{c}_N)\mathbf{d} = -\mathbf{c}_B\mathbf{y}_k + c_k = -z_k + c_k$$

But since $c_k - z_k < 0$ (because x_k was eligible to enter the basis), then $\mathbf{cd} < 0$, which is the necessary and sufficient condition for unboundedness. In Exercise 3.28 we ask the reader to verify that $\mathbf{d}$ given above is indeed an (extreme) direction of the feasible region.

Example 6.

$$\text{Minimize } -x_1 - 3x_2$$

$$\text{Subject to } x_1 - 2x_2 \leq 4$$

$$-x_1 + x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

The problem, illustrated in Figure 3.8, clearly has an unbounded optimal solution. After introducing the slack variables x_3 and x_4 , we get the constraint matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$. Now consider the basic feasible solution whose basis $\mathbf{B}$ is $[\mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

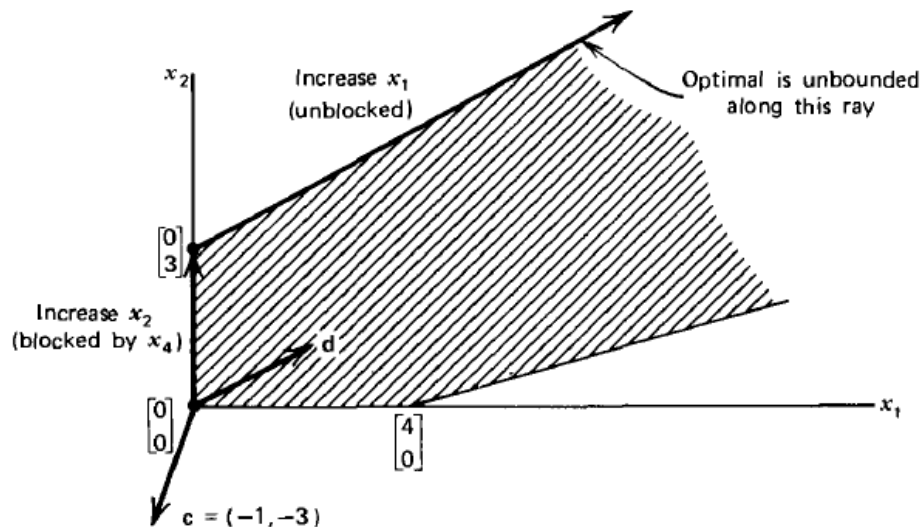


Figure 3.8. Unbounded optimal.

Calculate $z_1 - c_1$ and $z_2 - c_2$ as follows:

$$\begin{aligned} z_1 - c_1 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_1 - c_1 \\ &= (0, 0) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} z_2 - c_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\ &= (0, 0) \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 3 \\ &= 3 \end{aligned}$$

So we increase x_2 with the most positive $z_j - c_j$. Note that $\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{a}_2 x_2$, and hence

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

The maximum value of x_2 is 3, at which instant x_4 drops to zero. Therefore the new basic feasible solution is $(x_1, x_2, x_3, x_4) = (0, 3, 10, 0)$. The new basis $\mathbf{B}$ is $[\mathbf{a}_3, \mathbf{a}_2] = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ with inverse $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Calculate $z_1 - c_1$ and $z_4 - c_4$ as follows:

$$\begin{aligned} z_1 - c_1 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_1 - c_1 \\ &= (0, -3) \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1 \\ &= (0, -3) \begin{bmatrix} -1 \\ -1 \end{bmatrix} + 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} z_4 - c_4 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_4 - c_4 \\ &= (0, -3) \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 \\ &= (0, -3) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= -3 \end{aligned}$$

Note that $z_1 - c_1 > 0$ and $y_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore the optimal solution is unbounded. In this case, if x_1 is increased and x_4 is kept zero, we get the following solution:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{a}_1 x_1$$

$$\begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} x_1 = \begin{bmatrix} 10 + x_1 \\ 3 + x_1 \end{bmatrix}$$

$$x_4 = 0$$

Note that this solution is feasible for all $x_1 \geq 0$. In particular,

$$x_1 - 2x_2 + x_3 = x_1 - 2(3 + x_1) + (10 + x_1) = 4,$$

and

$$-x_1 + x_2 + x_4 = -x_1 + (3 + x_1) + 0 = 3$$

Furthermore, $z = -9 - 4x_1$, which approaches $-\infty$ as x_1 approaches ∞ . Therefore the optimal solution is unbounded by moving along the ray

$$\{(0, 3, 10, 0) + x_1(1, 1, 1, 0) : x_1 \geq 0\}$$

Again note that the necessary and sufficient condition for unboundedness holds, namely

$$\mathbf{cd} = (-1, -3, 0, 0) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = -4 < 0$$

2.4.5. The Simplex Algorithm (Minimization problem)

INITIALIZATION STEP

Choose a starting basic feasible solution with basis $\mathbf{B}$.

MAIN STEP

1. Solve the system $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ (with unique solution $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \bar{\mathbf{b}}$). Let $\mathbf{x}_B = \bar{\mathbf{b}}$, $\mathbf{x}_N = \mathbf{0}$, and $z = \mathbf{c}_B\mathbf{x}_B$.

2. Solve the system $\mathbf{wB} = \mathbf{c}_B$ (with unique solution $\mathbf{w} = \mathbf{c}_B \mathbf{B}^{-1}$). Calculate $z_j - c_j = \mathbf{w} \mathbf{a}_j - c_j$ for all nonbasic variables. Let

$$z_k - c_k = \text{Maximum}_{j \in R} z_j - c_j$$

where R is the current set of indices associated with the nonbasic variables. If $z_k - c_k \leq 0$, then stop with the current basic feasible solution as an optimal solution. Otherwise go to step 3.

3. Solve the system $\mathbf{B} \mathbf{y}_k = \mathbf{a}_k$ (with unique solution $\mathbf{y}_k = \mathbf{B}^{-1} \mathbf{a}_k$). If $\mathbf{y}_k \leq \mathbf{0}$, then stop with the conclusion that the optimal solution is unbounded along the ray

$$\left\{ \begin{bmatrix} \bar{\mathbf{b}} \\ \mathbf{0} \end{bmatrix} + x_k \begin{bmatrix} -\mathbf{y}_k \\ \mathbf{e}_k \end{bmatrix} : x_k \geq 0 \right\}$$

where $\mathbf{e}_k$ is an $n - m$ vector of zeros except for a 1 at the k th position. If $\mathbf{y}_k \not\leq \mathbf{0}$, go to step 4.

4. Here x_k enters the basis and the blocking variable x_{B_r} leaves the basis, where the index r is determined by the following minimum ratio test:

$$\frac{\bar{b}_r}{y_{rk}} = \text{Minimum}_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

Update the basis $\mathbf{B}$ where $\mathbf{a}_k$ replaces $\mathbf{a}_{B_r}$, the index set R and repeat step 1.

Example 1.

$$\begin{array}{ll} \text{Minimize} & -x_1 - 3x_2 \\ \text{Subject to} & 2x_1 + 3x_2 \leq 6 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

The problem is illustrated in Figure 3.9. After introducing the nonnegative slack variables x_3 and x_4 , we get the following constraints:

$$\begin{array}{rcl} 2x_1 + 3x_2 + x_3 & = & 6 \\ -x_1 + x_2 + x_4 & = & 1 \\ x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

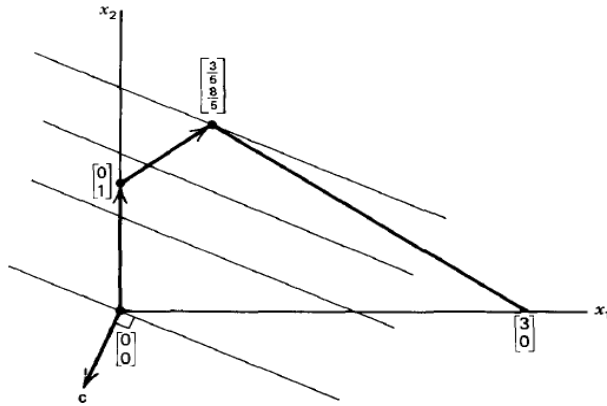


Figure 3.9: Example of simplex method

Iteration 1

Let $\mathbf{B} = [\mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{N} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$. Solving the system $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ leads to $x_{B_1} = x_3 = 6$ and $x_{B_2} = x_4 = 1$. The nonbasic variables

are x_1 and x_2 and the objective $z = \mathbf{c}_B \mathbf{x}_B = (0, 0) \begin{bmatrix} 6 \\ 1 \end{bmatrix} = 0$. In order to determine which variable enters the basis, calculate $z_j - c_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j = \mathbf{w} \mathbf{a}_j - c_j$. First we find $\mathbf{w}$ by solving the system $\mathbf{w} \mathbf{B} = \mathbf{c}_B$:

$$(w_1, w_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (0, 0) \Rightarrow w_1 = w_2 = 0$$

$$z_1 - c_1 = \mathbf{w} \mathbf{a}_1 - c_1 = 1$$

$$z_2 - c_2 = \mathbf{w} \mathbf{a}_2 - c_2 = 3$$

Therefore x_2 is increased. In order to determine x_2 we need to calculate y_2 by solving the system $\mathbf{B} \mathbf{y}_2 = \mathbf{a}_2$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow y_{12} = 3 \quad \text{and} \quad y_{22} = 1$$

The variable x_{B_1} leaving the basis is determined by the following minimum ratio test:

$$\text{Minimum} \left\{ \frac{\bar{b}_1}{y_{12}}, \frac{\bar{b}_2}{y_{22}} \right\} = \text{Minimum} \left\{ \frac{6}{3}, \frac{1}{1} \right\} = 1$$

Therefore the index $r = 2$; that is, $x_{B_2} = x_4$ leaves the basis. This is also obvious by noting that

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_2 \quad (3.11)$$

and x_4 first drops to zero when $x_2 = 1$.

Iteration 2

The variable x_2 enters the basis and x_4 leaves the basis:

$$\mathbf{B} = [\mathbf{a}_3, \mathbf{a}_2] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{N} = [\mathbf{a}_1, \mathbf{a}_4] = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

Now $\mathbf{x}_B$ can be determined by solving $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ or simply by noting that $x_2 = 1$ in Equation (3.11).

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The objective value is $z = -3$. Calculate $\mathbf{w}$ by $\mathbf{wB} = \mathbf{c}_B$:

$$(w_1, w_2) \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = (0, -3) \Rightarrow w_1 = 0 \quad \text{and} \quad w_2 = -3$$

$$z_1 - c_1 = \mathbf{w}\mathbf{a}_1 - c_1$$

$$= (0, -3) \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 = 4$$

The variable x_4 left the basis in the previous iteration and cannot enter the basis in this iteration since $z_4 - c_4 < 0$ (see Exercise 3.33). Therefore x_1 is increased. Solve the system $\mathbf{B}\mathbf{y}_1 = \mathbf{a}_1$:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow y_{11} = 5 \quad \text{and} \quad y_{21} = -1$$

Since $y_{21} < 0$, then $x_{B_1} = x_3$ leaves the basis as x_1 is increased. This is also clear by noting that

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ -1 \end{bmatrix} x_1$$

and x_3 drops to zero when $x_1 = \frac{3}{5}$.

Iteration 3

Here x_1 enters the basis and x_3 leaves the basis:

$$\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{N} = [\mathbf{a}_3, \mathbf{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{8}{5} \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The objective value $z = -\frac{27}{5}$. Calculate $\mathbf{w}$ by $\mathbf{wB} = \mathbf{c}_B$:

$$(w_1, w_2) \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = (-1, -3) \Rightarrow w_1 = -\frac{4}{5} \quad \text{and} \quad w_2 = -\frac{3}{5}$$

The variable x_3 left the basis in the last iteration and cannot enter the basis in this iteration (because $z_3 - c_3 < 0$).

$$z_4 - c_4 = \mathbf{wa}_4 - c_4$$

$$= \left(-\frac{4}{5}, -\frac{3}{5}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 = -\frac{3}{5}$$

Therefore $z_j - c_j < 0$ for all nonbasic variables, and the current point is optimal. The optimal solution is therefore

$$(x_1, x_2, x_3, x_4) = \left(\frac{3}{5}, \frac{8}{5}, 0, 0\right)$$

with objective value $-\frac{27}{5}$. Figure 3.9 displays the sequence of steps that the simplex method took to reach the optimal point.

2.4.4. The simplex method in Tableau format

At each iteration the following linear systems of equations need be solved: $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, $\mathbf{w}\mathbf{B} = \mathbf{c}_B$, and $\mathbf{B}\mathbf{y}_k = \mathbf{a}_k$. Various procedures for solving and updating these systems will lead to different algorithms that all lie under the general framework of the simplex method described above. In this section we describe the simplex method in tableau format. In subsequent chapters we shall describe several procedures for solving the preceding systems for general linear programs as well as for problems with special structure such as network flow problems.

Suppose that we have a starting basic feasible solution $\mathbf{x}$ with basis $\mathbf{B}$. The linear programming problem can be represented as follows.

$$\text{Minimize} \quad z$$

$$\text{Subject to} \quad z - \mathbf{c}_B\mathbf{x}_B - \mathbf{c}_N\mathbf{x}_N = 0 \quad (3.12)$$

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \quad (3.13)$$

$$\mathbf{x}_B, \quad \mathbf{x}_N \geq \mathbf{0}$$

From Equation (3.13) we have

$$\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \quad (3.14)$$

Multiplying (3.14) by $\mathbf{c}_B$ and adding to Equation (3.12), we get

$$z + \mathbf{0}\mathbf{x}_B + (\mathbf{c}_B\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_N)\mathbf{x}_N = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} \quad (3.15)$$

Currently $\mathbf{x}_N = \mathbf{0}$, and from Equations (3.14) and (3.15) we get $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}$. Also, from (3.14) and (3.15) we can conveniently represent the current basic feasible solution in the following tableau. Here we think of z as a (basic) variable to be minimized. The objective row will be referred to as row 0 and the remaining rows are rows 1 through m . The right-hand-side column

(RHS) will denote the values of the basic variables (including the objective function). The basic variables are identified on the far left column.

	z	x_B	x_N	RHS	
z	1	0	$c_B B^{-1} N - c_N$	$c_B B^{-1} b$	Row 0
x_B	0	I	$B^{-1} N$	$B^{-1} b$	Rows 1 through m

Not only does this tableau give us the value of the objective function $c_B B^{-1} b$ and the basic variables $B^{-1} b$ on the right-hand side, but it also gives us all the information we need to proceed with the simplex method. Actually the cost row gives us $c_B B^{-1} N - c_N$, which consists of the $z_j - c_j$'s for the nonbasic variables. So row zero will tell us if we are at the optimal solution (if each $z_j - c_j \leq 0$), and which nonbasic variable to increase otherwise. If x_k is increased, then the vector $y_k = B^{-1} a_k$, which is stored in the tableau in rows 1 through m under the variable x_k , will help us determine by how much x_k can be increased. If $y_k \leq 0$, then x_k can be increased indefinitely without being blocked, and the optimal objective is unbounded. On the other hand, if $y_k \not\leq 0$, that is, if y_k has at least one positive component, then the increase in x_k will be blocked by one of the current basic variables, which drops to zero. The minimum ratio test (which can be performed since $B^{-1} b = \bar{b}$ and y_k are both available in the tableau) determines the blocking variable. We would like to have a scheme that will do the following.

1. Update the basic variables and their values.
2. Update the $z_j - c_j$ values of the new nonbasic variables.
3. Update the y_j columns.

Pivoting

All of the foregoing tasks can be simultaneously accomplished by a simple pivoting operation. If x_k enters the basis and x_B leaves the basis, then pivoting on y_{rk} can be stated as follows.

1. Divide row r by y_{rk} .
2. For $i = 1, 2, \dots, m$ and $i \neq r$, update the i th row by adding to it $-y_{ik}$ times the new r th row.
3. Update row zero by adding to it $c_k - z_k$ times the new r th row. The two tableaux of Tables 3.1 and 3.2 represent the situation immediately before and after pivoting.

Table 3.1 Before Pivoting

	z	x_{B_1}	x_{B_r}	x_{B_m}	x_j	x_k	RHS
z	1	0	...	0	...	$z_j - c_j$	$c_B \bar{b}$
x_{B_1}	0	1	...	0	...	y_{1j}	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_r}	0	0	...	1	...	y_{rj}	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_m}	0	0	...	0	...	y_{mj}	$\bar{b}_m$

Table 3.2 After Pivoting

	z	x_{B_1}	x_{B_r}	x_{B_m}	x_k	RHS
z	1	0	...	$\frac{c_k - z_k}{y_{rk}}$	0	$c_B \bar{b} - (z_k - c_k) \frac{\bar{b}_r}{y_{rk}}$
					$-\frac{y_{rj}}{y_{rk}}(z_j - c_j)$	
x_{B_1}	0	1	...	$\frac{-y_{1k}}{y_{rk}}$	0	$\bar{b}_1 - \frac{y_{1k}}{y_{rk}} \bar{b}_r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k	0	0	...	$\frac{1}{y_{rk}}$	0	$\frac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_m}	0	0	...	$\frac{-y_{mk}}{y_{rk}}$	1	$\bar{b}_m - \frac{y_{mk}}{y_{rk}} \bar{b}_r$

Let us examine the implications of the pivoting operation.

1. The variable x_k entered the basis and x_{B_r} left the basis. This is illustrated on the left-hand side of the tableau by replacing x_{B_r} with x_k . For the purpose of the following iteration, the new x_{B_r} is now x_k .
2. The right-hand side of the tableau gives the current values of the basic variables (review Equation 3.7). The nonbasic variables are kept zero.
3. Suppose that the original columns of the new basic and nonbasic variables are $\hat{\mathbf{B}}$ and $\hat{\mathbf{N}}$ respectively. Through a sequence of elementary row operations (characterized by pivoting at the intermediate iterations), the original tableau reduces to the current tableau with $\hat{\mathbf{B}}$ replaced by $\mathbf{I}$. From Chapter 2 we know that this is equivalent to premultiplication by $\hat{\mathbf{B}}^{-1}$. Thus, pivoting results in a new tableau that gives the new $\hat{\mathbf{B}}^{-1}\hat{\mathbf{N}}$ under the nonbasic variables, an updated set of $z_j - c_j$'s for the new nonbasic variables, and the values of the new basic variables and objective function.

The Simplex Method in Tableau Format (Minimization Problem)**INITIALIZATION STEP**

Find an initial basic feasible solution with basis $\mathbf{B}$. Form the following initial tableau.

	z	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
z	1	$\mathbf{0}$	$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N$	$\mathbf{c}_B \bar{\mathbf{b}}$
$\mathbf{x}_B$	$\mathbf{0}$	$\mathbf{I}$	$\mathbf{B}^{-1} \mathbf{N}$	$\bar{\mathbf{b}}$

MAIN STEP

Let $z_k - c_k = \text{Maximum}\{z_j - c_j : j \in R\}$. If $z_k - c_k \leq 0$, then stop; the current solution is optimal. Otherwise examine $\mathbf{y}_k$. If $\mathbf{y}_k \leq \mathbf{0}$, then stop; the optimal solution is unbounded along the ray

$$\left\{ \begin{bmatrix} \mathbf{B}^{-1} \bar{\mathbf{b}} \\ \mathbf{0} \end{bmatrix} + x_k \begin{bmatrix} -\mathbf{y}_k \\ \mathbf{e}_k \end{bmatrix} : x_k \geq 0 \right\}$$

where $\mathbf{e}_k$ is a vector of zeros except for a 1 at the k th position. If $\mathbf{y}_k \not\leq \mathbf{0}$, determine the index r as follows:

$$\frac{\bar{b}_r}{y_{rk}} = \text{Minimum} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

Update the tableau by pivoting at y_{rk} . Update the basic and nonbasic variables where x_k enters the basis and x_{B_r} leaves the basis, and repeat the main step.

Example 1.

$$\text{Minimize } x_1 + x_2 - 4x_3$$

$$\text{Subject to } x_1 + x_2 + 2x_3 \leq 9$$

$$x_1 + x_2 - x_3 \leq 2$$

$$-x_1 + x_2 + x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

Introduce the nonnegative slack variables x_4 , x_5 , and x_6 . The problem becomes the following.

$$\text{Minimize } x_1 + x_2 - 4x_3 + 0x_4 + 0x_5 + 0x_6$$

$$\text{Subject to } x_1 + x_2 + 2x_3 + x_4 = 9$$

$$x_1 + x_2 - x_3 + x_5 = 2$$

$$-x_1 + x_2 + x_3 + x_6 = 4$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Since $\mathbf{b} \geq \mathbf{0}$, then we can choose our initial basis as $\mathbf{B} = [\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6] = \mathbf{I}_3$, and we indeed have $\mathbf{B}^{-1}\mathbf{b} = \bar{\mathbf{b}} \geq \mathbf{0}$. This gives the following initial tableau.

Iteration 1

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-1	-1	4	0	0	0	0
x_4	0	1	1	2	1	0	0	9
x_5	0	1	1	-1	0	1	0	2
x_6	0	-1	1	①	0	0	1	4

Iteration 2

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	3	-5	0	0	0	-4	-16
x_4	0	③	-1	0	1	0	-2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	-1	1	1	0	0	1	4

Iteration 3

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	-4	0	-1	0	-2	-17
x_1	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
x_5	0	0	2	0	0	1	1	6
x_3	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{13}{3}$

This is the optimal tableau since $z_j - c_j \leq 0$ for all nonbasic variables. The optimal solution is given by

$$x_1 = \frac{1}{3}, x_2 = 0, x_3 = \frac{13}{3}$$

$$z = -17$$

Note that the current optimal basis consists of the columns $\mathbf{a}_1$, $\mathbf{a}_5$, and $\mathbf{a}_3$, namely

$$\mathbf{B} = [\mathbf{a}_1, \mathbf{a}_5, \mathbf{a}_3] = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Interpretation of Entries in the Simplex Tableau

Consider the following typical simplex tableau and assume nondegeneracy.

	z	$\mathbf{x}_B$	$\mathbf{x}_N$	RHS
z	1	0	$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N$	$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{b}$
$\mathbf{x}_B$	0	I	$\mathbf{B}^{-1} \mathbf{N}$	$\mathbf{B}^{-1} \mathbf{b}$

The tableau may be thought of as a representation of both the basic variables $\mathbf{x}_B$ and the cost variable z in terms of the nonbasic variables $\mathbf{x}_N$. The nonbasic variables can therefore be thought of as independent variables, whereas $\mathbf{x}_B$ and z are dependent variables. From row zero we have

$$\begin{aligned} z &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} - (\mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N) \mathbf{x}_N \\ &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} + \sum_{j \in R} (c_j - z_j) x_j \end{aligned}$$

and hence the rate of change of z as a function of a typical nonbasic variable x_j , namely $\partial z / \partial x_j$, is simply $c_j - z_j$. In order to minimize z , we should increase x_j if $\partial z / \partial x_j < 0$, that is, if $z_j - c_j > 0$.

Also, the basic variables can be represented in terms of the nonbasic variables as follows:

$$\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N$$

$$\begin{aligned}
&= \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in R} \mathbf{B}^{-1}\mathbf{a}_j x_j \\
&= \mathbf{B}^{-1}\mathbf{b} - \sum_{j \in R} \mathbf{y}_j x_j
\end{aligned}$$

Therefore $\partial \mathbf{x}_B / \partial x_j = -\mathbf{y}_j$; that is, $-\mathbf{y}_j$ is the rate of change of the basic variables as a function of the nonbasic variable x_j . In other words, if x_j increases by 1 unit, then the i th basic variable x_{B_i} decreases by an amount y_{ij} , or simply $\partial x_{B_i} / \partial x_j = -y_{ij}$. A column $\mathbf{y}_j$ can be alternatively interpreted as follows. Recall that $\mathbf{B}\mathbf{y}_j = \mathbf{a}_j$, and hence $\mathbf{y}_j$ represents the linear combination of the basic columns that are needed to represent $\mathbf{a}_j$. More specifically,

$$\mathbf{a}_j = \sum_{i=1}^m \mathbf{a}_{B_i} y_{ij}$$

The simplex tableau also gives us a convenient way of predicting the rate of change of the objective function and the value of the basic variables as a function of the right-hand-side vector $\mathbf{b}$. Since the right-hand-side vector usually represents scarce resources, we can predict the rate of change of the objective function as the availability of the resources is varied. In particular,

$$z = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} - \sum_{j \in R} (z_j - c_j) x_j$$

and hence $\partial z / \partial \mathbf{b} = \mathbf{c}_B \mathbf{B}^{-1}$. If the original identity consists of slack variables with zero costs, then the elements of row zero at the final tableau under the slacks give $\mathbf{c}_B \mathbf{B}^{-1} \mathbf{I} - \mathbf{0} = \mathbf{c}_B \mathbf{B}^{-1}$, which is $\partial z / \partial \mathbf{b}$. More specifically, if we let $\mathbf{w} = \mathbf{c}_B \mathbf{B}^{-1}$, then $\partial z / \partial b_i = w_i$.

Similarly, the rate of change of the basic variables as a function of the right-hand-side vector $\mathbf{b}$ is given by

$$\frac{\partial \mathbf{x}_B}{\partial \mathbf{b}} = \mathbf{B}^{-1}$$

In particular, $\partial x_{B_i} / \partial \mathbf{b}$ is the i th row of $\mathbf{B}^{-1}$, $\partial \mathbf{x}_B / \partial b_j$ is the j th column of $\mathbf{B}^{-1}$, and $\partial x_{B_i} / \partial b_j$ is the (i, j) entry of $\mathbf{B}^{-1}$. If the tableau corresponds to a degenerate basic feasible solution, then as a nonbasic variable x_j increases, at least one of

the basic variables may become immediately negative (see Equation 3.5) destroying feasibility. In this case a change of basis is necessary to restore feasibility, leading to nondifferentiability of the objective value as a function of x_j . The advanced reader will note that, in this case, the quantities given in this section correspond to one-sided directional derivatives.

Identifying $\mathbf{B}^{-1}$ from the Simplex Tableau

The basis inverse matrix can be identified from the simplex tableau as follows. Assume that the original tableau has an identity matrix. The process of reducing the basis matrix $\mathbf{B}$ of the original tableau to an identity matrix in the current tableau, is equivalent to premultiplying rows 1 through m of the original tableau by $\mathbf{B}^{-1}$ to produce the current tableau (why?). This also converts the identity matrix of the original tableau to $\mathbf{B}^{-1}$. Therefore, $\mathbf{B}^{-1}$ can be extracted from the current tableau as the submatrix in rows 1 through m under the original identity columns.

Example 2. To illustrate the interpretation of simplex tableau, consider example 1 above at iteration 2. Then

$$\frac{\partial z}{\partial x_1} = -3, \quad \frac{\partial z}{\partial x_2} = 5, \quad \frac{\partial z}{\partial x_6} = 4$$

$$\frac{\partial x_4}{\partial x_1} = -3, \quad \frac{\partial x_5}{\partial x_1} = 0, \quad \frac{\partial x_3}{\partial x_6} = -1$$

$$\frac{\partial \mathbf{x}_B}{\partial x_2} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\frac{\partial z}{\partial b_1} = \frac{\partial z}{\partial b_2} = 0, \quad \frac{\partial z}{\partial b_3} = -4$$

$$\frac{\partial x_5}{\partial b_2} = 1, \quad \frac{\partial x_4}{\partial b_3} = -2$$

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The vector $\mathbf{a}_2$ can be represented as a linear combination of the basic columns as follows: $\mathbf{a}_2 = -1\mathbf{a}_4 + 2\mathbf{a}_5 + \mathbf{a}_3$. At iteration 3 we have

$$\frac{\partial z}{\partial x_2} = 4, \quad \frac{\partial z}{\partial x_6} = 2$$

$$\frac{\partial x_5}{\partial x_2} = -2, \quad \frac{\partial x_5}{\partial x_6} = -1, \quad \frac{\partial x_3}{\partial x_2} = -\frac{2}{3}$$

$$\frac{\partial \mathbf{x}_B}{\partial x_2} = \begin{bmatrix} \frac{1}{3} \\ -2 \\ -\frac{2}{3} \end{bmatrix}$$

$$\frac{\partial z}{\partial b_1} = -1, \quad \frac{\partial z}{\partial b_2} = 0, \quad \frac{\partial z}{\partial b_3} = -2$$

$$\frac{\partial \mathbf{x}_B}{\partial b_3} = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}$$

$$\frac{\partial x_1}{\partial b_1} = \frac{1}{3}, \quad \frac{\partial x_3}{\partial b_1} = \frac{1}{3}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{2}{3} \\ 0 & 1 & 1 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

The vector $\mathbf{a}_2$ can be represented as a linear combination of the basic columns as follows: $\mathbf{a}_2 = -\frac{1}{3}\mathbf{a}_1 + 2\mathbf{a}_5 + \frac{2}{3}\mathbf{a}_3$.

2.4.5. Finding a starting basic feasible solution

In the previous section, we develop the simplex method with the assumptions that an initial basic feasible solution is at hand. In many cases, such a solution is not readily available and some work may be needed to get the simplex method started. In this section we describe two procedures (the two-phase method and Big-M method), both involving artificial variables to obtain an initial basic feasible solution to a slightly modified set of constraints. The simplex method is used to eliminate the artificial variables and to solve the original problem. We also discuss in more detail the difficulties associated with degeneracy. In particular, we show that

the simplex method converges in a finite number of steps, even in the presence of degeneracy, provided that a special rule for exiting from basis is adopted.

2.4.5.1. The initial basic feasible solution

Recall that the simplex method starts with a basic feasible solution and moves to an improved basic feasible solution, until the optimal point is reached or else unboundedness of the objective function is verified. However, in order to initialize the simplex method, a basis $\mathbf{B}$ with $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$ must be available. We shall show that the simplex method can always be initiated with a very simple basis, namely the identity.

Easy Case

Suppose that the constraints are of the form $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an m nonnegative vector. By adding the slack vector $\mathbf{x}_s$, the constraints are put in the following standard form: $\mathbf{Ax} + \mathbf{x}_s = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x}_s \geq \mathbf{0}$. Note that the new $m \times (m + n)$ constraint matrix $(\mathbf{A}, \mathbf{I})$ has rank m , and a basic feasible solution of this system is at hand, by letting $\mathbf{x}_s = \mathbf{b}$ be the basic vector, and $\mathbf{x} = \mathbf{0}$ be the nonbasic vector. With this starting basic feasible solution, the simplex method can be applied.

Some Bad Cases

In many occasions, finding a starting basic feasible solution is not as straightforward as the case described above. To illustrate, suppose that the constraints are of the form $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ but the vector $\mathbf{b}$ is not nonnegative. In this case, after introducing the slack vector $\mathbf{x}_s$, we cannot let $\mathbf{x} = \mathbf{0}$, because $\mathbf{x}_s = \mathbf{b}$ violates the nonnegativity requirement.

Another situation occurs when the constraints are of the form $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, where $\mathbf{b} \geq \mathbf{0}$. After subtracting the slack vector $\mathbf{x}_s$, we get $\mathbf{Ax} - \mathbf{x}_s = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, and $\mathbf{x}_s \geq \mathbf{0}$. Again, there is no obvious way of picking a basis $\mathbf{B}$ from the matrix $(\mathbf{A}, -\mathbf{I})$ with $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$.

In general, any linear programming problem can be transformed into a problem of the following form.

Minimizes $\mathbf{cx}$

Subject to $\mathbf{Ax} = \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

where $\mathbf{b} \geq \mathbf{0}$ (if $b_i < 0$, the i th row can be multiplied by -1). This can be accomplished by introducing slack variables and by simple manipulation of the constraints and variables. If $\mathbf{A}$ contains an identity matrix, then an immediate basic feasible solution is at hand, by simply letting $\mathbf{B} = \mathbf{I}$, and since $\mathbf{b} \geq \mathbf{0}$, then $\mathbf{B}^{-1}\mathbf{b} = \mathbf{b} \geq \mathbf{0}$. Otherwise, something else must be done.

Example 1.

a. Consider the following constraints:

$$x_1 + 2x_2 \leq 4$$

$$-x_1 + x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

After adding the slack variables x_3 and x_4 , we get

$$x_1 + 2x_2 + x_3 = 4$$

$$-x_1 + x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

An obvious starting basic feasible solution is given by $\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and

$$\mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

b. Consider the following constraints:

$$x_1 + x_2 + x_3 \leq 6$$

$$-2x_1 + 3x_2 + 2x_3 \geq 3$$

$$x_2, x_3 \geq 0$$

Note that x_1 is unrestricted. So the change of variable $x_1 = x_1^+ - x_1^-$ is made. Also the slack variables x_4 and x_5 are introduced. This leads to the following constraints in standard form:

$$x_1^+ - x_1^- + x_2 + x_3 + x_4 = 6$$

$$-2x_1^+ + 2x_1^- + 3x_2 + 2x_3 - x_5 = 3$$

$$x_1^+, x_1^-, x_2, x_3, x_4, x_5 \geq 0$$

Note that the constraint matrix does not contain an identity and no obvious feasible basis **B** can be extracted.

c. Consider the following constraints:

$$x_1 + x_2 - 2x_3 \leq -3$$

$$-2x_1 + x_2 + 3x_3 \leq 7$$

$$x_1, x_2, x_3 \geq 0$$

Since the right-hand side of the first constraint is negative, the first inequality is multiplied by -1 . Introducing the slack variables x_4 and x_5 leads to the following system:

$$-x_1 - x_2 + 2x_3 - x_4 = 3$$

$$-2x_1 + x_2 + 3x_3 + x_5 = 7$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Note again that this constraint matrix contains no identity.

Artificial Variables

After manipulating the constraints and introducing slack variables, suppose that the constraints are put in the format $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b} \geq \mathbf{0}$ is an m vector. Further suppose that $\mathbf{A}$ has no identity submatrix (if $\mathbf{A}$ has an identity submatrix then we have an obvious starting basic feasible solution). In this case we shall resort to artificial variables to get a starting basic feasible solution, and then use the simplex method itself to get rid of these artificial variables.

To illustrate, suppose that we change the restrictions by adding an artificial vector $\mathbf{x}_a$ leading to the system $\mathbf{Ax} + \mathbf{x}_a = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x}_a \geq \mathbf{0}$. Note that by construction, we forced an identity matrix corresponding to the artificial vector.

This gives an immediate basic feasible solution of the new system, namely $x_a = \mathbf{b}$ and $\mathbf{x} = \mathbf{0}$. Even though we now have a starting basic feasible solution, and the simplex method can be applied, we have in effect changed the problem. In order to get back to our original problem, we must force these artificial variables to zero, because $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{Ax} + \mathbf{x}_a = \mathbf{b}$ with $\mathbf{x}_a = \mathbf{0}$. In other words, *artificial variables* are only a tool to get the simplex method started, but we must guarantee that these variables will eventually drop to zero.

At this stage, it is worthwhile to note the difference between slack and artificial variables. A slack variable is introduced to put the problem in equality form, and the slack variable can very well be positive, which means that the inequality holds as a strict inequality. *Artificial variables*, however, are not legitimate variables, and they may be introduced to facilitate the initiation of the simplex method. These artificial variables, however, must eventually drop to zero in order to attain feasibility in the original problem.

Example 2.

Consider the following constraints:

$$\begin{aligned}x_1 + 2x_2 &\geq 4 \\-3x_1 + 4x_2 &\geq 5 \\2x_1 + x_2 &\leq 6 \\x_1, x_2 &\geq 0\end{aligned}$$

Introducing the slack variables x_3 , x_4 , and x_5 , we get

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 4 \\-3x_1 + 4x_2 - x_4 &= 5 \\2x_1 + x_2 + x_5 &= 6 \\x_1, x_2, x_3, x_4, x_5 &\geq 0\end{aligned}$$

This constraint matrix has no identity submatrix. We can introduce three artificial variables to obtain a starting basic feasible solution. Note, however, that x_5 appears in the last row only, and it has coefficient 1. So we only need to introduce two artificial variables x_6 and x_7 , which leads to the following system.

<i>Legitimate variables</i>	<i>Artificial variables</i>	
$x_1 + 2x_2 - x_3$	$+ x_6$	$= 4$
$- 3x_1 + 4x_2$	$- x_4$	$+ x_7 = 5$
$2x_1 + x_2$	$+ x_5$	$= 6$
$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$		

Now we have an immediate starting basic feasible solution of the new system, namely $x_5 = 6$, $x_6 = 4$, and $x_7 = 5$. The rest of the variables are nonbasic and have value zero. Needless to say, we eventually would like for the artificial variables x_6 and x_7 to drop to zero.

A) The Two-phase method

There are various methods that can be used to eliminate the artificial variables. One of these methods is to minimize the sum of the artificial variables, subject to the constraints $\mathbf{Ax} + \mathbf{x}_a = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x}_a \geq \mathbf{0}$. If the original problem has a feasible solution, then the optimal value of this problem is zero, where all the artificial variables drop to zero. More importantly, as the artificial variables drop to zero, they leave the basis, and legitimate variables enter instead. Eventually all artificial variables leave the basis (this is not always the case, because we may have an artificial variable in the basis at level zero; this will be discussed later in greater detail). The basis then consists of legitimate variables. In other words, we get a basic feasible solution of the original system $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, and the simplex method can be started with the original objective $\mathbf{cx}$. If, on the other hand, after solving this problem we have a positive artificial variable, then the original problem has no feasible solutions (why?). This procedure is called the *two-phase method*. In the first phase we reduce artificial variables to value zero, or conclude that the original problem has no feasible solutions. In the former case, the second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the phase I. The two-phase method is outlined below.

Phase I

Solve the following linear program starting with the basic feasible solution $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_a = \mathbf{b}$.

$$\text{Minimize } \mathbf{1x}_a$$

$$\text{Subject to } \mathbf{Ax} + \mathbf{x}_a = \mathbf{b}$$

$$\mathbf{x}, \mathbf{x}_a \geq \mathbf{0}$$

If at optimality $\mathbf{x}_a \neq \mathbf{0}$, then stop; the original problem has no feasible solutions. Otherwise let the basic and nonbasic legitimate variables be $\mathbf{x}_B$ and $\mathbf{x}_N$. (We are assuming that all artificial variables left the basis. The case where some artificial variables remain in the basis at zero level will be discussed later.) Go to phase II.

Phase II

Solve the following linear program starting with the basic feasible solution $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $\mathbf{x}_N = \mathbf{0}$.

$$\text{Minimize} \quad \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N$$

$$\text{Subject to} \quad \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$

$$\mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0}$$

The optimal solution of the original problem is given by the optimal solution of the foregoing problem.

Example 3.

$$\text{Minimize} \quad x_1 - 2x_2$$

$$\text{Subject to} \quad x_1 + x_2 \geq 2$$

$$-x_1 + x_2 \geq 1$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

The feasible region and the path taken by phase I and phase II to reach the optimal point are shown in Figure 4.1. After introducing the slack variables x_3, x_4, x_5 , the following problem is obtained.

$$\text{Minimize} \quad x_1 - 2x_2$$

$$\text{Subject to} \quad x_1 + x_2 - x_3 = 2$$

$$-x_1 + x_2 - x_4 = 1$$

$$x_2 + x_5 = 3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

An initial identity is not available. So introduce the artificial variables x_6 and x_7 (note that the last constraint does not need an artificial variable). Phase I starts by minimizing $x_0 = x_6 + x_7$.

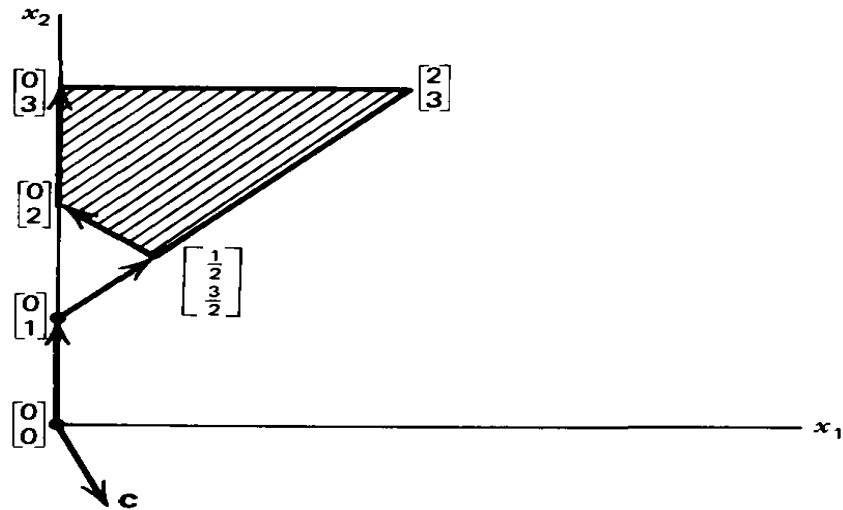


Figure 3.6.1 Example of two-phase method
PHASE I

x_0	x_1	x_2	x_3	x_4	x_5	ARTIFICIALS		RHS
						x_6	x_7	
1	0	0	0	0	0	-1	-1	0
0	1	1	-1	0	0	1	0	2
0	-1	1	0	-1	0	0	1	1
0	0	1	0	0	1	0	0	3

Add rows 1 and 2 to row 0 so that $z_6 - c_6 = z_7 - c_7 = 0$ will be displayed.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0	1	0	2	-1	-1	0	0	0	3
x_6	0	1	1	-1	0	0	1	0	2
x_7	0	-1	①	0	-1	0	0	1	1
x_5	0	0	1	0	0	1	0	0	3

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0	1	2	0	-1	1	0	0	-2	1
x_6	0	②	0	-1	1	0	1	-1	1
x_2	0	-1	1	0	-1	0	0	1	1
x_5	0	1	0	0	1	1	0	-1	2

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
x_0	1	0	0	0	0	0	-1	-1	0
x_1	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
x_5	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

This is the end of phase I. We have a starting basic feasible solution, $(x_1, x_2) = (\frac{1}{2}, \frac{3}{2})$. Now we are ready to start phase II, where the original objective is minimized starting from the extreme point $(\frac{1}{2}, \frac{3}{2})$ (see Figure 4.1). The artificial variables are disregarded from any further consideration.

PHASE II

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-1	2	0	0	0	0
0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$

Multiply rows 1 and 2 by 1 and -2 respectively and add to row 0 producing $z_1 - c_1 = z_2 - c_2 = 0$.

z	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	$\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{5}{2}$
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
x_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$

z	x_1	x_2	x_3	x_4	x_5	RHS
1	-3	0	2	0	0	-4
x_4	0	2	-1	1	0	1
x_2	0	1	-1	0	0	2
x_5	0	-1	1	0	1	1

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-1	0	0	0	-2	-6
x_4	0	1	0	0	1	1	2
x_2	0	0	1	0	0	1	3
x_3	0	-1	0	1	0	1	1

Since $z_j - c_j \leq 0$ for all nonbasic variables, the optimal point $(0, 3)$ with objective -6 is reached. Note that phase I moved from the infeasible point $(0, 0)$, to the infeasible point $(0, 1)$, and finally to the feasible point $(\frac{1}{2}, \frac{3}{2})$. From this extreme point, phase II moved to the feasible point $(0, 2)$, and finally to the optimal point $(0, 3)$. This is illustrated in Figure 4.1. The purpose of phase I is to get us to an extreme point of the feasible region, while phase II takes us from this feasible point to the optimal point.

Analysis of the Two-Phase Method

At the end of phase I either $x_a \neq 0$ or else $x_a = 0$. These two cases are discussed in detail below.

Case A: $x_a \neq 0$

If $x_a \neq 0$, then the original problem has no feasible solutions, because if there is an $x \geq 0$ with $Ax = b$, then $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is a feasible solution of the phase I problem and $0(x) + 1(0) = 0 < 1x_a$, violating optimality of x_a .

Example 4. (Empty Feasible Region)

$$\text{Minimize } -3x_1 + 4x_2$$

$$\text{Subject to } x_1 + x_2 \leq 4$$

$$2x_1 + 3x_2 \geq 18$$

$$x_1, x_2 \geq 0$$

The constraints admit no feasible points, as shown in Figure 4.2. This will be detected by phase I. Introducing the slack variables x_3 and x_4 , we get the following constraints in standard form:

$$x_1 + x_2 + x_3 = 4$$

$$2x_1 + 3x_2 - x_4 = 18$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Since no convenient basis exists, introduce the artificial variable x_5 into the second constraint. Phase I is used to get rid of the artificial.

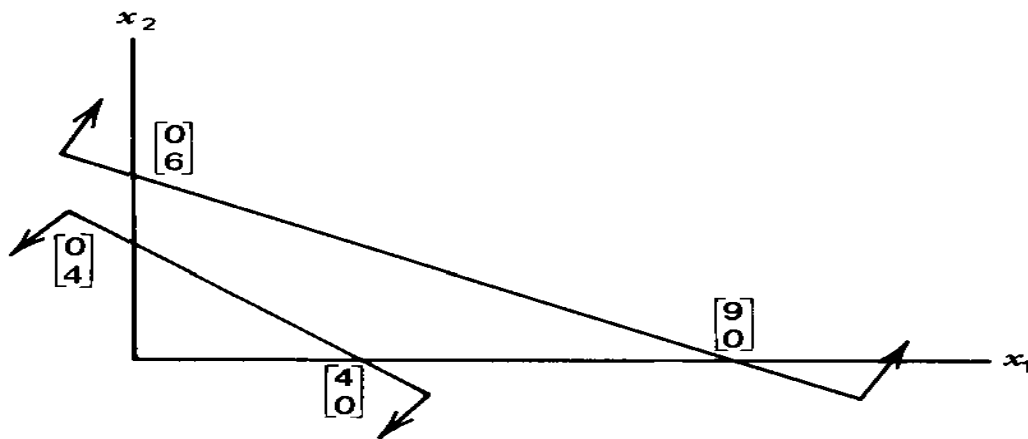


Figure 3.6.2 Empty feasible region

PHASE I

x_0	x_1	x_2	x_3	x_4	x_5	RHS
1	0	0	0	0	-1	0
0	1	1	1	0	0	4
0	2	3	0	-1	1	18

Add row 2 to row 0 so that $z_5 - c_5 = 0$ is displayed.

	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	2	3	0	-1	0	18
x_3	0	1	①	1	0	0	4
x_5	0	2	3	0	-1	1	18

	x_0	x_1	x_2	x_3	x_4	x_5	RHS
x_0	1	-1	0	-3	-1	0	6
x_2	0	1	1	1	0	0	4
x_5	0	-1	0	-3	-1	1	6

The optimality criterion of the simplex method, namely $z_j - c_j \leq 0$, holds for all variables; but the artificial $x_5 > 0$. We conclude that the original problem has no feasible solutions.

B) The Big-M method

Recall that artificial variables constitute a tool that can be used to initiate the simplex method. However, the presence of artificial variables at a positive level, by construction, means that the current point is an infeasible solution of the original system. The two-phase method is one way to get rid of the artificials.

However, during phase I of the two-phase method the original cost coefficients are essentially ignored. Phase I of the two-phase method seeks any basic feasible solution, not necessarily a good one. Another possibility for eliminating the artificial variables is to assign coefficients for these variables in the original objective function in such a way as to make their presence in the basis at a positive level, very unattractive from the objective function point of view. To illustrate, suppose that we want to solve the following linear programming problem, where $\mathbf{b} \geq \mathbf{0}$.

$$\text{Minimize } \mathbf{c}\mathbf{x}$$

$$\text{Subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

If no convenient basis is known, we introduce the artificial vector $\mathbf{x}_a$, which leads to the following system:

$$\mathbf{A}\mathbf{x} + \mathbf{x}_a = \mathbf{b}$$

$$\mathbf{x}, \mathbf{x}_a \geq \mathbf{0}$$

The starting basic feasible solution is given by $\mathbf{x}_a = \mathbf{b}$ and $\mathbf{x} = \mathbf{0}$. In order to reflect the undesirability of a nonzero artificial vector, the objective function is modified such that a large penalty is paid for any such solution. More specifically consider the following problem.

$$\text{Minimize } \mathbf{c}\mathbf{x} + M\mathbf{1}\mathbf{x}_a$$

$$\text{Subject to } \mathbf{A}\mathbf{x} + \mathbf{x}_a = \mathbf{b}$$

$$\mathbf{x}, \mathbf{x}_a \geq \mathbf{0}$$

where M is a very large positive number. The term $M\mathbf{1}\mathbf{x}_a$ can be interpreted as a penalty to be paid by any solution with $\mathbf{x}_a \neq \mathbf{0}$. Even though the starting solution $\mathbf{x} = \mathbf{0}$, $\mathbf{x}_a = \mathbf{b}$ is feasible to the new constraints, it has a very unattractive objective value, namely $M\mathbf{1}\mathbf{b}$. Therefore the simplex method itself will try to get the artificial variables out of the basis, and then continue to find the optimal solution of the original problem.

The big- M method is illustrated by the following numerical example.

Example 5.

$$\text{Minimize } x_1 - 2x_2$$

$$\text{Subject to } x_1 + x_2 \geq 2$$

$$-x_1 + x_2 \geq 1$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

This example was solved earlier by the two-phase method (Example 4.3). The slack variables x_3, x_4, x_5 are introduced and the artificial variables x_6 and x_7 are incorporated in the first two constraints. The modified objective function is $z = x_1 - 2x_2 + Mx_6 + Mx_7$, where M is a large positive number. This leads to the following sequence of tableaux.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
	1	-1	2	0	0	0	-M	-M	0
	0	1	1	-1	0	0	1	0	2
	0	-1	1	0	-1	0	0	1	1
	0	0	1	0	0	1	0	0	3

Multiply rows 1 and 2 by M and add to row 0.

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	-1	$2+2M$	$-M$	$-M$	0	0	0	$3M$
x_6	0	1	1	-1	0	0	1	0	2
x_7	0	-1	①	0	-1	0	0	1	1
x_5	0	0	1	0	0	1	0	0	3

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	$1+2M$	0	$-M$	$2+M$	0	0	$-2-2M$	$-2+M$
x_6	0	②	0	-1	1	0	1	-1	1
x_2	0	-1	1	0	-1	0	0	1	1
x_5	0	1	0	0	1	1	0	-1	2

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	0	$\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}-M$	$-\frac{3}{2}-M$	$-\frac{5}{2}$
x_1	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
x_2	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
x_5	0	0	0	$\frac{1}{2}$	1	1	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	-3	0	2	0	0	$-2-M$	$-M$	-4
x_4	0	2	0	-1	1	0	1	-1	1
x_2	0	1	1	-1	0	0	1	0	2
x_5	0	-1	0	1	0	1	-1	0	1

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	-1	0	0	0	-2	$-M$	$-M$	-6
x_4	0	1	0	0	1	1	0	-1	2
x_2	0	0	1	0	0	1	0	0	3
x_3	0	-1	0	1	0	1	-1	0	1

Since $z_j - c_j \leq 0$ for each nonbasic variable, the last tableau gives the optimal solution. The sequence of points generated in the (x_1, x_2) space is illustrated in Figure 4.3.

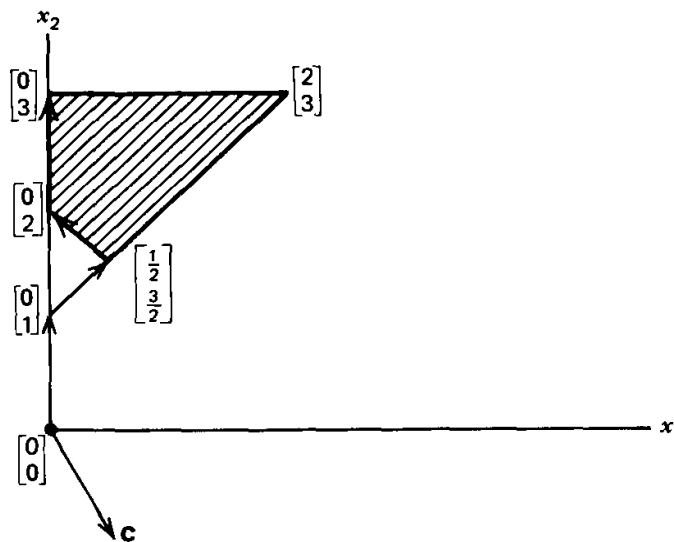


Figure 3.6.3 Example of the Big-M method

Analysis of the Big-M Method

We now discuss in more detail the possible cases that may arise while solving the big- M problem. The original problem P and the big- M problem $P(M)$ are stated below, where the vector $\mathbf{b}$ is nonnegative.

$$\begin{aligned}
 \text{Problem } P: \quad & \text{Minimize} \quad \mathbf{c} \mathbf{x} \\
 & \text{Subject to} \quad \mathbf{A} \mathbf{x} = \mathbf{b} \\
 & \quad \quad \quad \mathbf{x} \geq \mathbf{0} \\
 \\
 \text{Problem } P(M): \quad & \text{Minimize} \quad \mathbf{c} \mathbf{x} + M \mathbf{1} \mathbf{x}_a \\
 & \text{Subject to} \quad \mathbf{A} \mathbf{x} + \mathbf{x}_a = \mathbf{b} \\
 & \quad \quad \quad \mathbf{x}, \mathbf{x}_a \geq \mathbf{0}
 \end{aligned}$$

Since problem $P(M)$ has a feasible solution (say $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_a = \mathbf{b}$), then while solving it by the simplex method one of the following two cases may arise.

1. We shall arrive at an optimal solution of $P(M)$.
2. Conclude that $P(M)$ has an unbounded optimal solution, that is, $z \rightarrow -\infty$.

Of course, we are interested in conclusions about problem P and not $P(M)$. The following analysis will help us to draw such conclusions.

Case A: Finite Optimal Solution of $P(M)$.

Under this case, we have two possibilities. First, the optimal solution to $P(M)$ has all artificials at value zero, and second, not all artificials are zero. These cases are discussed below.

Subcase A1: $(\mathbf{x}^*, \mathbf{0})$ is an optimal solution of $P(M)$.

In this case $\mathbf{x}^*$ is an optimal solution to problem P . This can be easily verified as follows. Let $\mathbf{x}$ be any feasible solution to problem P , and note that $(\mathbf{x}, \mathbf{0})$ is a feasible solution to problem $P(M)$. Since $(\mathbf{x}^*, \mathbf{0})$ is an optimal solution of problem $P(M)$, then $\mathbf{c} \mathbf{x}^* + 0 \leq \mathbf{c} \mathbf{x} + 0$, that is, $\mathbf{c} \mathbf{x}^* \leq \mathbf{c} \mathbf{x}$. Since $\mathbf{x}^*$ is a feasible solution of problem P , then $\mathbf{x}^*$ is indeed an optimal solution of P . This case was illustrated by Example 4.6 above.

Subcase A2: $(\mathbf{x}^*, \mathbf{x}_a^*)$ is an optimal solution of $P(M)$ and $\mathbf{x}_a^* \neq \mathbf{0}$.

If M is a very large positive number, then we conclude that there exists no feasible solution of P . To illustrate this case, suppose on the contrary that $\mathbf{x}$ was

a feasible solution of P . Then $(x, 0)$ would be a feasible solution of $P(M)$ and by optimality of (x^*, x_a^*) we have

$$cx^* + M1x_a^* \leq cx + 0 = cx$$

Since M is very large, $x_a^* \geq 0$ and $x_a^* \neq 0$, and since x_a^* corresponds to one of the finite number of basic feasible solutions, the preceding inequality is impossible (why?). Therefore x could not have been a feasible solution of P . A formal

Example 5. (No feasible solution)

$$\text{Minimize } -x_1 - 3x_2 + x_3$$

$$\text{Subject to } x_1 + x_2 + 2x_3 \leq 4$$

$$-x_1 + x_3 \geq 4$$

$$x_3 \geq 3$$

$$x_1, x_2, x_3 \geq 0$$

Introduce the slack variables x_4, x_5 , and x_6 . Also, introduce the two artificial variables x_7 and x_8 for the last two constraints with cost coefficients equal to M . This leads to the following sequence of tableaux.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
1	1	3	-1	0	0	0	$-M$	$-M$	0
0	1	1	2	1	0	0	0	0	4
0	-1	0	1	0	-1	0	1	0	4
0	0	0	1	0	0	-1	0	1	3

Multiply rows 2 and 3 by M and add to row 0.

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	$1-M$	3	$-1+2M$	0	$-M$	$-M$	0	$7M$
x_4	0	1	1	②	1	0	0	0	4
x_7	0	-1	0	1	0	-1	0	1	4
x_8	0	0	0	1	0	0	-1	0	3

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	$\frac{3}{2} - 2M$	$\frac{7}{2} - M$	0	$\frac{1}{2} - M$	$-M$	$-M$	0	$2+3M$
x_3	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	2
x_7	0	$-\frac{3}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-1	0	1	2
x_8	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	-1	0	1

Since $M > 0$ is very large, then $z_j - c_j \leq 0$ for all nonbasic variables; that is, the simplex optimality criteria hold. Since the artificials x_7 and x_8 still remain in the basis at a positive level, then we conclude that the original problem has no feasible solutions.

Case B: $P(M)$ has an unbounded optimal solution, that is, $z \rightarrow -\infty$.

Suppose that during the solution of the big- M problem, the updated column y_k is ≤ 0 , where the index k is that of the *most positive* $z_j - c_j$. Then problem $P(M)$ has an unbounded optimal solution. In the meantime, if all artificials are equal to zero, then the original problem has an unbounded optimal solution. Otherwise, if at least one artificial variable is positive, then the original problem is infeasible. These two subcases are discussed in more detail below.

Subcase B1 : $z_k - c_k = \text{Maximum } (z_i - c_i) > 0$, $y_k \leq 0$, and all artificials are equal to zero.

In this case we have a feasible solution of the original problem. Furthermore, since problem $P(M)$ is unbounded, then there is a direction, $(d_1, d_2) \geq (0, 0)$ of the set $\{(x, x_a) : Ax + x_a = b, x \geq 0, x_a \geq 0\}$ such that $cd_1 + M1d_2 < 0$ (recall that this is the necessary and sufficient condition for unboundedness of $P(M)$). Since M is very large and $d_2 \geq 0$, then $cd_1 + M1d_2 < 0$ implies that $d_2 = 0$ and hence $cd_1 < 0$. Thus we have found a direction d_1 of the set $\{x : Ax = b, x \geq 0\}$ such that $cd_1 < 0$ (why?). This implies that problem P has an unbounded optimal solution.

Example 6.

$$\text{Minimize } -x_1 - x_2$$

$$\text{Subject to } x_1 - x_2 - x_3 = 1$$

$$-x_1 + x_2 + 2x_3 - x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Introduce the artificial variables x_5 and x_6 with cost coefficient M . This leads to the following sequence of tableaux.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
1	1	1	0	0	$-M$	$-M$	0
0	1	-1	-1	0	1	0	1
0	-1	1	2	-1	0	1	1

Multiply rows 1 and 2 by M and add to row 0.

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	M	$-M$	0	0	$2M$
x_5	0	1	-1	0	1	0	1
x_6	0	-1	1	(2)	0	1	1

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	$1 + \frac{1}{2}M$	$1 - \frac{1}{2}M$	0	$-\frac{1}{2}M$	0	$\frac{3}{2}M$
x_5	0	($\frac{1}{2}$)	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
x_3	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$

z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	2	0	$-M-2$	$-M-1$	-3
x_1	0	1	-1	0	2	1	3
x_3	0	0	0	1	1	1	2

The most positive $z_j - c_j$ corresponds to x_2 and $y_2 \leq 0$. Therefore the big- M problem is unbounded. Furthermore, the artificial variables x_5 and x_6 are equal to zero, and hence the original program has an unbounded optimal solution along the ray $\{(3, 0, 2, 0) + \lambda(1, 1, 0, 0) : \lambda \geq 0\}$.

CHAPTER THREE: Duality Theory and further variations of the simplex method

3.1. Dual linear programs

For every linear program there is another associated linear program. This new linear program satisfies some very important properties. It may be used to obtain the solution to the original program. Its variables provide extremely useful information about the optimal solution to the original linear program. To distinguish points of view in this and subsequent chapters we shall call the original linear programming problem the *primal* (linear programming) problem.

We shall begin by formulating this new linear program, called the *dual* (linear programming) problem, and proceed to develop some of its important properties. These properties will lead to two new algorithms, the dual simplex method and the primal-dual algorithm, for solving linear programs. Finally, we shall discuss the effect of variation in the data, that is, the cost coefficients, the right-hand-side coefficients, and the constraint coefficients on the optimal solution to a linear program.

3.1.1. Formulation of the dual problem

Associated with each linear programming problem there is another linear programming problem called the dual. The dual linear program possesses many

important properties relative to the original primal linear program. There are two important forms (definitions) of duality: the *canonical* form of duality and the *standard* form of duality. These two forms are completely equivalent.

Canonical Form of Duality

Suppose that the primal linear program is given in the form:

$$\begin{aligned} \text{P: Minimize } & \mathbf{cx} \\ \text{Subject to } & \mathbf{Ax} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Then the *dual* linear program is defined by:

$$\begin{aligned} \text{D: Maximize } & \mathbf{wb} \\ \text{Subject to } & \mathbf{wA} \leq \mathbf{c} \\ & \mathbf{w} \geq \mathbf{0} \end{aligned}$$

Note that there is exactly one dual variable for each primal constraint and exactly one dual constraint for each primal variable. We shall say more about this later.

Example 1.

Consider the following linear program and its dual.

$$\text{P: Minimize } 6x_1 + 8x_2$$

$$\text{Subject to } 3x_1 + x_2 \geq 4$$

$$5x_1 + 2x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

$$\text{D: Maximize } 4w_1 + 7w_2$$

$$\text{Subject to } 3w_1 + 5w_2 \leq 6$$

$$w_1 + 2w_2 \leq 8$$

$$w_1, w_2 \geq 0$$

Before proceeding further, try solving both problems and compare their optimal objective values. This will provide a hint of things to come.

In the canonical definition of duality it is important for problem P to have a “Minimization” objective with all “greater than or equal to” constraints and all “nonnegative” variables. In theory, to apply the canonical definition of duality we must first convert the primal linear program to the foregoing format. However, in practice it is possible to immediately write down the dual of any linear program. We shall discuss this shortly.

Standard Form of Duality

Another equivalent definition of duality applies when the constraints are equalities. Suppose that the primal linear program is given in the form:

$$\text{P: Minimize } \mathbf{c}\mathbf{x}$$

$$\text{Subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

Then the *dual* linear program is defined by:

$$\text{D: Maximize } \mathbf{wb}$$

$$\text{Subject to } \mathbf{wA} \leq \mathbf{c}$$

$$\mathbf{w} \text{ unrestricted}$$

Example 2.

Consider the following linear program and its dual.

$$\text{P: Minimize } 6x_1 + 8x_2$$

$$\text{Subject to } 3x_1 + x_2 - x_3 = 4$$

$$5x_1 + 2x_2 - x_4 = 7$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$\text{D: Maximize } 4w_1 + 7w_2$$

$$\text{Subject to } 3w_1 + 5w_2 \leq 6$$

$$w_1 + 2w_2 \leq 8$$

$$-w_1 \leq 0$$

$$-w_2 \leq 0$$

$$w_1, w_2 \text{ unrestricted}$$

Given one of the definitions, canonical or standard, it is easy to demonstrate that the other definition is valid. For example, suppose that we accept the standard form as a definition and wish to demonstrate that the canonical form is correct. By adding slack variables to the canonical form of a linear program, we may apply the standard form of duality to obtain the dual problem.

$$\text{P: Minimize } \mathbf{cx}$$

$$\text{D: Maximize } \mathbf{wb}$$

$$\text{Subject to } \mathbf{Ax} - \mathbf{Ix}_s = \mathbf{b}$$

$$\mathbf{x}, \mathbf{x}_s \geq \mathbf{0}$$

$$\text{Subject to } \mathbf{wA} \leq \mathbf{c}$$

$$-\mathbf{wI} \leq \mathbf{0}$$

$$\mathbf{w} \text{ unrestricted}$$

But since $-\mathbf{w}\mathbf{I} \leq \mathbf{0}$ is the same as $\mathbf{w} \geq \mathbf{0}$, we obtain the canonical form of the dual problem.

Dual of the Dual

Since the dual linear program is itself a linear program, we may wonder what its dual might be. Consider the dual in canonical form:

$$\begin{aligned} &\text{Maximize } \mathbf{wb} \\ &\text{Subject to } \mathbf{wA} \leq \mathbf{c} \\ &\quad \mathbf{w} \geq \mathbf{0} \end{aligned}$$

Applying the transformation techniques of Chapter 1, we may rewrite this problem in the form:

$$\begin{aligned} &\text{Minimize } (-\mathbf{b}')\mathbf{w}' \\ &\text{Subject to } (-\mathbf{A}')\mathbf{w}' \geq (-\mathbf{c}') \\ &\quad \mathbf{w}' \geq \mathbf{0} \end{aligned}$$

The dual linear program for this linear program is given by (letting $\mathbf{x}'$ play the role of the row vector of dual variables):

$$\begin{aligned} &\text{Maximize } \mathbf{x}'(-\mathbf{c}') \\ &\text{Subject to } \mathbf{x}'(-\mathbf{A}') \leq (-\mathbf{b}') \\ &\quad \mathbf{x}' \geq \mathbf{0} \end{aligned}$$

But this is the same as

$$\begin{aligned} &\text{Minimize } \mathbf{cx} \\ &\text{Subject to } \mathbf{Ax} \geq \mathbf{b} \\ &\quad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

which is precisely the original primal problem. Thus we have the following lemma.

Lemma 1

The dual of the dual is the primal.

This lemma indicates that the definitions may be applied in reverse. The terms “primal” and “dual” are relative to the frame of reference we choose.

Mixed Forms of Duality

In practice, many linear programs contain some constraints of the “less than or equal to” type, some of the “greater than or equal to” type and some of the “equal to” type. Also, variables may be “ ≥ 0 ,” “ ≤ 0 ,” or “unrestricted.” In theory, this presents no problem since we may apply the transformation techniques of Chapter 1 to convert any “mixed” problem to one of the primal or dual forms discussed above, after which the dual can be readily obtained. In practice such conversions can be tedious. Fortunately, it is not necessary actually to make these conversions, and it is possible to give immediately the dual of any linear program.

Consider the following linear program.

$$\text{Minimize } \mathbf{c}\mathbf{x}$$

$$\text{Subject to } \mathbf{A}_1\mathbf{x} \geq \mathbf{b}_1$$

$$\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$$

$$\mathbf{A}_3\mathbf{x} \leq \mathbf{b}_3$$

$$\mathbf{x} \geq \mathbf{0}$$

Converting this problem to the standard format, we get

$$\text{Minimize } \mathbf{c}\mathbf{x}$$

$$\text{Subject to } \mathbf{A}_1\mathbf{x} - \mathbf{I}\mathbf{x}_s = \mathbf{b}_1$$

$$\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$$

$$\mathbf{A}_3\mathbf{x} + \mathbf{I}\mathbf{x}_t = \mathbf{b}_3$$

$$\mathbf{x}, \mathbf{x}_s, \mathbf{x}_t \geq \mathbf{0}$$

The dual of this problem is

$$\text{Maximize } \mathbf{w}_1\mathbf{b}_1 + \mathbf{w}_2\mathbf{b}_2 + \mathbf{w}_3\mathbf{b}_3$$

$$\text{Subject to } w_1 A_1 + w_2 A_2 + w_3 A_3 \leq c$$

$$- w_1 I \leq 0$$

$$w_3 I \leq 0$$

$$w_1, w_2, w_3 \text{ unrestricted}$$

From this example we see that “greater than or equal to” constraints in the minimization problem give rise to “ ≥ 0 ” variables in the maximization problem. Also, “equal to” constraints in the minimization problem give rise to “unrestricted” variables in the maximization problem; and “less than or equal to” constraints in the minimization problem give rise to “ ≤ 0 ” variables in the maximization problem. In Exercise 6.5 we ask the reader to consider the various cases for variables and constraints in the minimization problem and their counterparts in the maximization problem. The complete results may be summarized in Table 6.1.

Table 6.1 Relationships Between Primal and Dual Problems

Variables	MINIMIZATION PROBLEM		MAXIMIZATION PROBLEM		Constraints
	≥ 0	$\longleftrightarrow$	$\leq$		
	≤ 0	$\longleftrightarrow$	$\geq$		
	Unrestricted	$\longleftrightarrow$	$=$		
Constraints	$\geq$	$\longleftrightarrow$	≥ 0	Variables	
	$\leq$	$\longleftrightarrow$	≤ 0		
	$=$	$\longleftrightarrow$	Unrestricted		

We may utilize this table to develop the dual of any linear program without first transforming it to the standard or canonical forms.

Example 3.

Consider the following linear program.

$$\text{Maximize } 8x_1 + 3x_2$$

$$\text{Subject to } x_1 - 6x_2 \geq 2$$

$$5x_1 + 7x_2 = -4$$

$$x_1 \leq 0$$

$$x_2 \geq 0$$

Applying the results of the table, we can immediately write down the dual.

$$\begin{aligned}
 &\text{Minimize} && 2w_1 - 4w_2 \\
 &\text{Subject to} && w_1 + 5w_2 \leq 8 \\
 &&& -6w_1 + 7w_2 \geq 3 \\
 &&& w_1 \leq 0 \\
 &&& w_2 \text{ unrestricted}
 \end{aligned}$$

3.2. Primal-Dual relationships

The definition we have selected for the dual problem leads to many important relationships between the primal and dual linear programs.

The Relationship Between Objective Values

Consider the canonical form of duality and let x_0 and w_0 be feasible solutions to the primal and dual programs respectively. Then $Ax_0 \geq b$, $x_0 \geq 0$, $w_0A \leq c$, and $w_0 \geq 0$. Multiplying $Ax_0 \geq b$ on the left by $w_0 \geq 0$ and $w_0A \leq c$ on the right by $x_0 \geq 0$, we get

$$cx_0 \geq w_0Ax_0 \geq w_0b$$

The result is the following.

Lemma 2

The objective function value for any feasible solution to the minimization problem is always greater than or equal to the objective function value for any feasible solution to the maximization problem. In particular, the objective value of any feasible solution of the minimization problem gives an upper bound on the optimal objective of the maximization problem. Similarly, the objective value of any feasible solution of the maximization problem is a lower bound on the optimal objective of the minimization problem.

The following corollaries are immediate consequences of Lemma 2.

Corollary 1

If $\mathbf{x}_0$ and $\mathbf{w}_0$ are feasible solutions to the primal and dual problems such that $\mathbf{c}\mathbf{x}_0 = \mathbf{w}_0\mathbf{b}$, then $\mathbf{x}_0$ and $\mathbf{w}_0$ are optimal solutions to their respective problems.

Corollary 2

If either problem has an unbounded objective value, then the other problem possesses no feasible solution.

This corollary indicates that unboundedness in one problem implies infeasibility in the other problem. Is this property symmetric? Does infeasibility in one problem imply unboundedness in the other? The answer is “not necessarily.” This is best illustrated by the following example.

Example 4.

Consider the following primal and dual problems.

P: Minimize $-x_1 - x_2$

Subject to $x_1 - x_2 \geq 1$

$-x_1 + x_2 \geq 1$

$x_1, x_2 \geq 0$

D: Maximize $w_1 + w_2$

Subject to $w_1 - w_2 \leq -1$

$-w_1 + w_2 \leq -1$

$w_1, w_2 \geq 0$

Upon graphing both problems (in Figure 6.1) we find that neither problem possesses a feasible solution.

Theorem 1 (Fundamental Theorem of Duality)

With regard to the primal and dual linear programming problems, exactly one of the following statements is true.

1. Both possess optimal solutions $\mathbf{x}^*$ and $\mathbf{w}^*$ with $\mathbf{c}\mathbf{x}^* = \mathbf{w}^*\mathbf{b}$.
2. One problem has unbounded objective value, in which case the other problem must be infeasible.
3. Both problems are infeasible.

From this theorem we see that duality is not completely symmetric. The best we can say is that (here optimal means finite optimal, and unbounded means having an unbounded optimal objective):

P	OPTIMAL	$\Leftrightarrow$	D	OPTIMAL
P	UNBOUNDED	$\Rightarrow$	D	INFEASIBLE
D	UNBOUNDED	$\Rightarrow$	P	INFEASIBLE
P	INFEASIBLE	$\Rightarrow$	D	UNBOUNDED OR INFEASIBLE
D	INFEASIBLE	$\Rightarrow$	P	UNBOUNDED OR INFEASIBLE

3.3. The Dual Simplex method

In this section we describe the *dual simplex* method, which solves the dual problem directly on the (primal) simplex tableau. At each iteration we move from a basic feasible solution of the dual problem to an improved basic feasible solution until optimality of the dual (and also the primal) is reached, or else until we conclude that the dual is unbounded and that the primal is infeasible.

Interpretation of Dual Feasibility on the Primal Simplex Tableau

Consider the following linear programming problem.

Minimize $\mathbf{c}\mathbf{x}$

Subject to $\mathbf{A}\mathbf{x} \geq \mathbf{b}$

$\mathbf{x} \geq \mathbf{0}$

Let $\mathbf{B}$ be a basis that is not necessarily feasible and consider the following tableau.

		SLACK VARIABLES							RHS
	z	x_1	x_2	$\dots$	x_n	x_{n+1}	$\dots$	x_{n+m}	
z	1	$z_1 - c_1$	$z_2 - c_2$	$\dots$	$z_n - c_n$	$z_{n+1} - c_{n+1}$	$\dots$	$z_{n+m} - c_{n+m}$	$\mathbf{c}_B \bar{\mathbf{b}}$
x_{B_1}	0	y_{11}	y_{12}	$\dots$	y_{1n}	$y_{1,n+1}$	$\dots$	$y_{1,n+m}$	$\bar{b}_1$
x_{B_2}	0	y_{21}	y_{22}	$\dots$	y_{2n}	$y_{2,n+1}$	$\dots$	$y_{2,n+m}$	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{B_m}	0	y_{m1}	y_{m2}	$\dots$	y_{mn}	$y_{m,n+1}$	$\dots$	$y_{m,n+m}$	$\bar{b}_m$

The tableau presents a primal feasible solution if $\bar{b}_i \geq 0$ for $i = 1, 2, \dots, m$; that is, if $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$. Furthermore, the tableau is optimal if $z_j - c_j \leq 0$ for $j = 1, 2, \dots, n + m$. Define $\mathbf{w} = \mathbf{c}_B \mathbf{B}^{-1}$. For $j = 1, 2, \dots, n$ we have

$$z_j - c_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j = \mathbf{w} \mathbf{a}_j - c_j$$

Hence $z_j - c_j \leq 0$ for $j = 1, 2, \dots, n$ implies that $\mathbf{w} \mathbf{a}_j - c_j \leq 0$ for $j = 1, 2, \dots, n$, which in turn implies that $\mathbf{w} \mathbf{A} \leq \mathbf{c}$. Furthermore, note that $\mathbf{a}_{n+i} = -\mathbf{e}_i$ and $c_{n+i} = 0$ for $i = 1, 2, \dots, m$ and so we have

$$\begin{aligned} z_{n+i} - c_{n+i} &= \mathbf{w} \mathbf{a}_{n+i} - c_{n+i} \\ &= \mathbf{w}(-\mathbf{e}_i) - 0 \\ &= -w_i \quad i = 1, 2, \dots, m \end{aligned}$$

In addition, if $z_{n+i} - c_{n+i} \leq 0$ for $i = 1, 2, \dots, m$, then $w_i \geq 0$ for $i = 1, 2, \dots, m$ and so $\mathbf{w} \geq \mathbf{0}$. We have just shown that $z_j - c_j \leq 0$ for $j = 1, 2, \dots, n + m$ implies that $\mathbf{w} \mathbf{A} \leq \mathbf{c}$ and $\mathbf{w} \geq \mathbf{0}$, where $\mathbf{w} = \mathbf{c}_B \mathbf{B}^{-1}$. In other words, dual feasibility is precisely the simplex optimality criteria $z_j - c_j \leq 0$ for all j . At optimality $\mathbf{w}^* = \mathbf{c}_B \mathbf{B}^{-1}$ and the dual objective $\mathbf{w}^* \mathbf{b} = (\mathbf{c}_B \mathbf{B}^{-1}) \mathbf{b} = \mathbf{c}_B (\mathbf{B}^{-1} \mathbf{b}) = \mathbf{c}_B \bar{\mathbf{b}} = z^*$; that is, the primal and dual objectives are equal. Thus we have the following result.

Lemma 4

At optimality of the primal minimization problem in the canonical form (that is, $z_j - c_j \leq 0$ for all j), $\mathbf{w}^* = \mathbf{c}_B \mathbf{B}^{-1}$ is an optimal solution to the dual problem. Furthermore $w_i^* = -(z_{n+i} - c_{n+i}) = -z_{n+i}$ for $i = 1, 2, \dots, m$.

The Dual Simplex Method

Consider the following linear programming problem.

$$\text{Minimize } \mathbf{c}\mathbf{x}$$

$$\text{Subject to } \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

In certain instances it is difficult to find a starting basic solution that is feasible (that is, all $\bar{b}_i \geq 0$) to a linear program without adding artificial variables. In these same instances it is often possible to find a starting basic, but not necessarily feasible, solution that is dual feasible (that is, all $z_j - c_j \leq 0$ for a minimization problem). In such cases it is useful to develop a variant of the simplex method that would produce a series of simplex tableaux that maintain dual feasibility and complementary slackness and strive toward primal feasibility.

	z	x_1	$\cdots$	x_j	$\cdots$	x_k	$\cdots$	x_n	RHS
z	1	$z_1 - c_1$	$\cdots$	$z_j - c_j$	$\cdots$	$z_k - c_k$	$\cdots$	$z_n - c_n$	$\mathbf{c}_B \bar{\mathbf{b}}$
x_{B_1}	0	y_{11}	$\cdots$	y_{1j}	$\cdots$	y_{1k}	$\cdots$	y_{1n}	$\bar{b}_1$
x_{B_2}	0	y_{21}	$\cdots$	y_{2j}	$\cdots$	y_{2k}	$\cdots$	y_{2n}	$\bar{b}_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{B_r}	0	y_{r1}	$\cdots$	y_{rj}	$\cdots$	y_{rk}	$\cdots$	y_{rn}	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_{B_m}	0	y_{m1}	$\cdots$	y_{mj}	$\cdots$	y_{mk}	$\cdots$	y_{mn}	$\bar{b}_m$

Consider the above tableau representing a basic solution at some iteration. Suppose that the tableau is dual feasible (that is, $z_j - c_j \leq 0$ for a minimization problem). Then, if the tableau is also primal feasible (that is, all $\bar{b}_i \geq 0$) then we have the optimal solution. Otherwise, consider some $\bar{b}_r < 0$. By selecting row r as a pivot row and some column k such that $y_{rk} < 0$ as a pivot column we can make the new right-hand side $\bar{b}_r' > 0$. Through a series of such pivots we hope to make all $\bar{b}_i \geq 0$ while maintaining all $z_j - c_j \leq 0$ and thus achieve optimality. The question that remains is how do we select the pivot column so as to maintain dual feasibility after pivoting. The pivot column k is determined by the

following minimum ratio test.

$$\frac{z_k - c_k}{y_{rk}} = \text{minimum}_j \left\{ \frac{z_j - c_j}{y_{rj}} : y_{rj} < 0 \right\} \quad (6.1)$$

Note that the new entries in row 0 after pivoting are given by:

$$(z_j - c_j)' = (z_j - c_j) - \frac{y_{rj}}{y_{rk}} (z_k - c_k)$$

If $y_{rj} \geq 0$, and since $z_k - c_k \leq 0$ and $y_{rk} < 0$, then $(y_{rj}/y_{rk})(z_k - c_k) \geq 0$ and hence $(z_j - c_j)' \leq z_j - c_j$. Since the previous solution was dual feasible, then $z_j - c_j \leq 0$ and hence $(z_j - c_j)' \leq 0$. Now consider the case where $y_{rj} < 0$. By 6.1 we have:

$$\frac{z_k - c_k}{y_{rk}} \leq \frac{z_j - c_j}{y_{rj}}$$

Multiplying both sides by $y_{rj} < 0$, we get $(z_j - c_j) - (y_{rj}/y_{rk})(z_k - c_k) \leq 0$, that is, $(z_j - c_j)' \leq 0$. To summarize, if the pivot column is chosen according to Equation (6.1), then the new basis obtained by pivoting at y_{rk} is still dual feasible. Moreover, the dual objective after pivoting is given by $\mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} - (z_k - c_k) \bar{b}_r / y_{rk}$. Since $z_k - c_k \leq 0$, $\bar{b}_r < 0$, and $y_{rk} < 0$, then $-(z_k - c_k) \bar{b}_r / y_{rk} \geq 0$ and the dual objective improves over the current value of $\mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} = \mathbf{wb}$.

We have just described a procedure that moves from a dual basic feasible solution to an improved (at least not worse) basic dual feasible solution. To complete the analysis we must consider the case when $y_{rj} \geq 0$ for all j and hence no column is eligible to be the pivot column. In this case the i th row reads: $\sum_j y_{rj} x_j = \bar{b}_r$. Since $y_{rj} \geq 0$ for all j and x_j is required to be nonnegative, then $\sum_j y_{rj} x_j \geq 0$ for any feasible solution. However, $\bar{b}_r < 0$. This contradiction shows that the primal is infeasible and the dual is unbounded (why?). In Exercise 6.31 we ask the reader to show directly that the dual is unbounded by constructing a direction satisfying the unboundedness criterion.

Summary of the Dual Simplex Method (Minimization Problem)

INITIALIZATION STEP

Find a basis $\mathbf{B}$ of the primal such that $z_j - c_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j \leq 0$ for all j (in Section 6.6 we describe a procedure for finding such a basis if it is not immediately available).

MAIN STEP

1. If $\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$, stop; the current solution is optimal. Otherwise select the pivot row r with $\bar{b}_r < 0$, say $\bar{b}_r = \text{Minimum} \{ \bar{b}_i \}$.
2. If $y_{rj} \geq 0$ for all j , stop; the dual is unbounded and the primal is infeasible. Otherwise select the pivot column k by the following minimum ratio test:

$$\frac{z_k - c_k}{y_{rk}} = \text{Minimum}_j \left\{ \frac{z_j - c_j}{y_{rj}} : y_{rj} < 0 \right\}$$

3. Pivot at y_{rk} and return to step 1.

Example 1.

Consider the following problem.

$$\text{Minimize } 2x_1 + 3x_2 + 4x_3$$

$$\text{Subject to } x_1 + 2x_2 + x_3 \geq 3$$

$$2x_1 - x_2 + 3x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

A starting basic solution that is dual feasible can be obtained by utilizing the slack variables x_4 and x_5 . This results from the fact that the cost vector is nonnegative. Applying the dual simplex method, we obtain the following series of tableaux.

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	-3	-4	0	0	0
x_4	0	-1	-2	-1	1	0	-3
x_5	0	-2	1	-3	0	1	-4

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-4	-1	0	-1	4
x_4	0	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$-\frac{9}{5}$	$-\frac{8}{5}$	$-\frac{1}{5}$	$\frac{28}{5}$
x_2	0	0	1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
x_1	0	1	0	$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$

Since $\bar{\mathbf{b}} \geq \mathbf{0}$ and $z_j - c_j \leq 0$ for all j , the optimal primal and dual solutions are at hand. In particular,

$$(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = \left(\frac{11}{5}, \frac{2}{5}, 0, 0, 0\right)$$

$$(w_1^*, w_2^*) = \left(\frac{8}{5}, \frac{1}{5}\right)$$

Note that w_1^* and w_2^* are respectively the negatives of the $z_j - c_j$ entries under the slack variables x_4 and x_5 . Also note that in each subsequent tableau the value of the objective function is increasing, as it should, for the dual (maximization) problem.